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ADMISSIBLE SETS AND THE SATURATION OF STRUCTURES

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0. Introduction

This paper is intended as a set of variations on a theme introduced by Ressayre [16].

For the moment, fix a countable admissible set A. In [16], Ressayre studied structures satisfying a model-theoretic property, slightly stronger than realising all possible types Σ-definable on A. These structures he called Σ-saturated; they will be called $\mathbf{A}$-saturated below. Among many other things, he proved the following: let the language L of $\mathfrak{M}$ be A-finite, let $|\mathfrak{M}|$ be A-finite, and suppose that L contains a constant $f(m)$ for each $m \in M$ such that $f \in A$ and $f(m)^{\mathfrak{M}} = m$; let $A(\mathfrak{M})$ be the primitive recursive closure of $A \cup \{\mathfrak{M}\}$; then $\mathfrak{M}$ is Σ-saturated if and only if whenever $c \in A(\mathfrak{M})$, X is Δ_0 on $A(\mathfrak{M})$, and

$$\forall x \in c\, \exists y \in A(\mathfrak{M})\, \exists z \in A \quad Xxyz,$$

there are $a \in A$, $b \in A(\mathfrak{M})$, such that

$$\forall x \in c\, \exists y \in b\, \exists z \in a \quad Xxyz.$$

This condition amounts to satisfying the set S of collection axioms of [16]. Sets satisfying this last collection property will be called $\mathbf{A}$-special below.

In fact, the structure $\langle A(\mathfrak{M}), \in, A\rangle$ has a property which we christen $+$-admissibility in Section 3 and analyse there. We generalise much of the theory of admissible sets to $+$-admissible structures $\mathscr{A}$, including Ressayre's work. We define the notion of $\mathscr{A}$-saturated structure, and of $\mathscr{A}$-special extension of $\mathscr{A}$. One useful consequence of this is that there is an elegant theory of structures Σ-saturated with respect to a given admissible set A and a fixed structure $\mathfrak{M}$ Σ-saturated with respect to A; for these will be simply the structures which are $\mathscr{B}$-saturated where $\mathscr{B} = \langle A(\mathfrak{M}), \in, A\rangle$. We remark in passing that Ressayre's relation-universality theorem for Σ-saturated structures, in this more general setting, becomes a fairly simple consequence of his existence theorem, making both simple consequences of the omitting-types theorem. A more detailed discussion appears in Section 4.

In [7] Friedman proves that if T is Σ on A, includes KP, and has a model endextending A, then it has a model whose wellfounded part contains the same ordinals as A. We show in Section 4 that the conclusion may be strengthened to: T has a model whose well-founded part is $\mathbf{A}$-special. This follows from a "truncation" property proved in Section 4: the well-founded part of an $\mathbf{A}$-saturated endextension of A is $\mathbf{A}$-special.

In Section 5 we show that certain forcing extensions of A are $\mathbf{A}$-special and apply this to show that there are admissible A, B such that B includes A, contains the same ordinals as A, but is not $\mathbf{A}$-special.

We work throughout in a set theory with urelements, and usually with structures whose universes are sets of urelements. In Section 2, we consider the behaviour of $A(\mathfrak{M})$ when the universe of $\mathfrak{M}$ is a set of urelements disjoint from A. Thus the membership relation on the universe of $\mathfrak{M}$, which is not part of $\mathfrak{M}$ qua structure, cannot get in the way. In Sections 2 and 4 we generalise theorems of Nadel and Stavi [14] characterising the pure sets in HYP ($\mathfrak{M}$) (a special case of $A(\mathfrak{M})$) to $A(\mathfrak{M})$.

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1. Preliminaries

Although it could be avoided, it will be convenient in this work to assume an underlying set theory which admits a proper class of urelements. Otherwise, the set-theoretic definitions and notations will be standard. The universe of all sets and urelements is V; the class of urelements is U. The variables w, x, y, z (possibly with subscripts) shall be used to denote members of V; the variables a, $b, c, \ldots, A, B, C, \ldots X, Y, Z$ (also possibly with subscripts) shall range over sets.

On is the class of ordinals; ω is the first nonzero limit ordinal. If A is a class, χ_A, the characteristic function of A, takes the value 0 on members of A, and the value 1 on nonmembers of A.

A set a is transitive, Trans (a), if each set which is an element of a is included in a. The transitive closure of x, TC (x), is the least transitive set y such that $x \subseteq y$. $\mathrm{rk}(x)$ is the set-theoretic rank of x, where $\mathrm{rk}(x)=0$ for $x \in U$, and $\mathrm{rk}(a)= \sup\{\mathrm{rk}(y)+1 : y \in a\}$ if a is a set.

If M is a set of urelements, V_M, the universe built up from M, is the union of the sequence of sets defined by transfinite induction as follows:

$$V_M(0)=M; \qquad V_M(\alpha+1)=M\cup P(V_M(\alpha)),$$
$$V_M(\lambda)=\bigcup\{V_M(\zeta): \zeta<\lambda\} \quad \text{for } \lambda \text{ a limit ordinal.}$$

$U(a)$ is $a\cap U$, for a a set. An element of V_0 is called a pure set. The pure part of a set A is $A\cap V_0$.

x is hereditarily finite (hereditarily countable) if TC (x) is finite (countable). HF is the class of hereditarily finite sets and urelements; HC is the class of hereditarily countable sets and urelements.

We shall require some facts about primitive recursive set functions (cf. [9]). Let $X_1, \dots, X_k$ be functions on the universe V. Then a function $F: V^n \to V$ for some $n\in\omega$ is primitive recursive in $X_1, \dots, X_k$ (written PR $(X_1, \dots, X_k)$) if it can be obtained from the initial functions described below by composition and recursion as described below:

(A) Initial Functions

(1) $F(x)=X_i(x) \quad 1\leqslant i\leqslant k$

(2) $F(x_1, \dots, x_n)=x_i \quad 1\leqslant n\in\omega, \quad 1\leqslant i\leqslant n$

(3) $F(x)=0$

(4) $F(x, y)=\{x, y\}$

(5) $F(x)=\begin{cases}\bigcup x & \text{if } x \text{ is a set of sets}\\ 0 & \text{otherwise}\end{cases}$

(6) $F(x)=\chi_U(x)=\begin{cases}0 & \text{if } x \text{ is an urelement}\\ 1 & \text{if } x \text{ is a set}\end{cases}$

(7) $C(x, y, u, v)=\begin{cases}x & \text{if } u\in v\\ y & \text{if } \neg u\in v\end{cases}$

(B) Composition

(1) If $m, n\in\omega$ and $G: V^{m+n+1}\to V$ and $H: V^n\to V$ are PR $(X_1, \dots, X_k)$, so is F defined by

$$F(x_1, \dots, x_n, y_1, \dots, y_m)=G(x_1, \dots, x_n, H(x_1, \dots, x_n), y_1, \dots, y_m).$$

(2) If $m, n\in\omega$ and $G: V^{m+1}\to V$ and $H: V^n\to V$ are PR $(X_1, \dots, X_k)$, so is F defined by

$$F(x_1, \dots, x_n, y_1, \dots, y_m)=G(H(x_1, \dots, x_n), y_1, \dots, y_m).$$

(C) Recursion

If $n\in\omega$ and $G: V^{n+2}\to V$ is PR $(X_1, \dots, X_k)$, then so is F, where F is the function satisfying

$$F(z, x_1, \dots, x_n)=G(\bigcup\{F(u, x_1, \dots, x_n): u\in z\}, z, x_1, \dots, x_n)$$

for all $x_1, \dots, x_n, z\in V$.

We remark that if G is $\mathrm{PR}(X_1, \ldots, X_k)$, then so is the function F which satisfies

$$F(z, x_1, \ldots, x_n) = G(\{\langle u, F(u, x_1, \ldots, x_n)\rangle : u \in \mathrm{TC}(z)\}, x_1, \ldots, x_n).$$

This can be seen by using (C) to directly define from G a function H which will satisfy

$$H(z, x_1, \ldots, x_n) = \{\langle u, F(u, x_1, \ldots, x_n)\rangle : u \in \mathrm{TC}(z) \cup \{z\}\}.$$

We say a relation on V is $\mathrm{PR}(X_1, \ldots, X_k)$ if its characteristic function is $\mathrm{PR}(X_1, \ldots, X_k)$.

A function or relation is PR if it is PR in the empty sequence of functions. Note that if $F: V^n \to V$ is PR, and $x_1, \ldots, x_n \in V_M$, then $F(x_1, \ldots, x_n) \in V_M$. In particular, F maps pure sets to pure sets.

If A is a set, let $\mathbf{A}$ be the structure $\langle A, \in \cap A^2, U(A)\rangle$.

If R is a class and $x \in V$, we define $L^R(x)$, the R-constructible sets from x, as the union of the sequence of sets defined as follows:

$$L_0^R(x) = \mathrm{TC}(x) \cup \{x\}$$

$$L_\lambda^R(x) = \bigcup \{L_\xi^R(x) : \xi < \lambda\} \quad \text{if } \lambda \text{ is a limit ordinal}$$

$$L_{\alpha+1}^R(x) = L_\alpha^R(x) \cup \mathrm{Def}_\alpha^R(L_\alpha^R(x))$$

where $\mathrm{Def}_\alpha^R(C)$, for any set C, is the set of sets definable on $(\mathbf{C}, R \cap C)$ (with parameters from C), by a first-order formula of length less than α. (This restriction on the length of the formula is of course vacuous once $\alpha \geq \omega$.) Note also that because we may have urelements, even a transitive C need not be contained in $\mathrm{Def}_\omega^R(C)$, since this latter is a set of sets.

If $R = 0$ or $x = 0$, it will be dropped from the notation, giving rise to such notation as L_α^R, $L_\alpha(x)$, and L_α.

We note finally that the function F satisfying

$$F(y, x) = 0 \quad \text{if } y \text{ is not an ordinal}$$

$$F(\alpha, x) = L_\alpha^R(x) \quad \text{for } \alpha \in \mathrm{On}$$

is $\mathrm{PR}(\chi_R)$. To see this, it is enough to verify that $\mathrm{Def}_\alpha^R(x)$ is $\mathrm{PR}(\chi_R)$ as a function of α and x. We leave this to the reader. A proof may be easily adapted from the appendices of [9].

As terms, formulas, etc., shall be considered as elements of admissible sets, it is necessary to give a precise definition of syntactical notions. A relation symbol of degree $n > 0$, for $n \in \omega$, is an ordered pair $\langle 0, \langle n, x\rangle\rangle$ for some $x \in V$. The relation symbol $\langle 0, \langle 2, 0\rangle\rangle$ is distinguished and will always be written "$=$". A function symbol of degree n is an ordered pair $\langle 1, \langle n, x\rangle\rangle$ for some $x \in V$. A constant symbol is a function symbol of degree 0. A variable is a pair $v_{ij} = \langle 2, \langle i, j\rangle\rangle$ for some $i, j \in \omega$. The double indexing of variables is a technical device which shall be useful below. Where it is no help, we shall use only the variables $v_n = v_{0n}$ for $n \in \omega$.

We shall use P, Q to range over relation symbols, O to range over function (operation) symbols, u, v to range over variables, all possibly with subscripts.

Terms are defined as usual, with the proviso that the term $Ot_1, \ldots, t_n$ is the set $\langle 3, \langle O, t_1, \ldots, t_n\rangle\rangle$. An atomic formula is a sequence $Pt_1, \ldots, t_n = \langle 4, \langle P, t_1, \ldots, t_n\rangle\rangle$ where P is a relation symbol of degree n and $t_1, \ldots, t_n$ are terms.

$\mathscr{L}_{\infty\omega}$ is the least collection containing all atomic formulas and closed under formation of negations, where $\neg\varphi = \langle 5, \varphi\rangle$, universal quantifications, where $\forall v_{ij}\varphi = \langle 6, i, j, \varphi\rangle$, and (possibly infinitary) disjunctions $\bigvee Y = \langle 7, Y\rangle$ provided only finitely many variables are free in the result. We shall let φ, ψ, θ range over $\mathscr{L}_{\infty\omega}$. A formula containing no variables freely is a sentence.

The quantifier rank of φ, $\mathrm{qr}(\varphi)$, is defined by induction on φ as follows:

$$\mathrm{qr}(\varphi) = 0 \quad \text{if } \varphi \text{ is atomic;} \qquad \mathrm{qr}(\neg\varphi) = \mathrm{qr}(\varphi);$$
$$\mathrm{qr}(\bigvee Y) = \sup\{\mathrm{qr}(\varphi) : \varphi \in Y\}; \qquad \mathrm{qr}(\forall v_n\varphi) = \mathrm{qr}(\varphi) + 1.$$

$\exists v_n$, $\bigwedge$, $\wedge$, $\vee$, and $\rightarrow$ are defined as usual. Ocasionally we may write $\exists u, v\varphi$ to abbreviate $\exists u\exists v\varphi$. We may also write $\exists u\varphi$ for $\exists u_1 \ldots u_n\varphi$; at such times, we shall specify the variables $u_1, \ldots, u_n$ in the accompanying text. $\mathbf{T}$ is the sentence $\forall v_0(v_0 = v_0)$. $\mathbf{F}$ is $\neg\mathbf{T}$. For $x \in V$, $\hat{x}$ is the individual constant $\langle 1, \langle 0, x\rangle\rangle$.

An alphabet is a set of relation symbols and function symbols. If K is an alphabet, $\mathscr{L}_{\infty\omega}(K)$ is the set of formulas of $\mathscr{L}_{\infty\omega}$ in which appear only relation and function symbols from K.

With the definitions given above, all the familiar syntactical relations and operations are primitive recursive. Wherever a specific instance of this fact is needed, we shall make a note of it.

A K-structure is a pair $\mathfrak{M} = \langle M, \rho\rangle$, where M is a set and ρ is a function defined on K which "interprets" predicate and function symbols by relations and operations of the appropriate degree. $\mathfrak{M}$ may also be written $\langle M, P^{\mathfrak{M}}, O^{\mathfrak{M}}\rangle_{P,O\in K}$. M, the universe of $\mathfrak{M}$, may be written $|\mathfrak{M}|$. "=" must always be interpreted by the identity relation.

We shall use notations such as $(\mathfrak{M}, R_i, G_j)_{i\in I, j\in J}$ (where for $i \in I$, R_i is a relation on M, and for $j \in J$, G_j is an operation on M) or $(\mathfrak{M}, \mathfrak{N})$ (when $|\mathfrak{N}| \subseteq |\mathfrak{M}|$) for the obvious expansions of $\mathfrak{M}$. When such notations appear, a suitable choice of a language for the expansion will always be possible.

For K-structures $\mathfrak{M}$ and $\mathfrak{N}$, we define as usual the notion $\mathfrak{M} \subseteq \mathfrak{N}$ ("$\mathfrak{N}$ is an extension of $\mathfrak{M}$") and the reduct $\mathfrak{M}\restriction K'$ of $\mathfrak{M}$ to a smaller language K'.

If φ is a formula all free variables of which are among the v_{ij} for $i \leqslant n$, $j \leqslant m$, and for each $i \leqslant n$, $s_i \in M^m$, then $\mathfrak{M} \models \varphi[s_0, \ldots, s_n]$, "$s_0, \ldots, s_n$ satisfy φ in $\mathfrak{M}$" is defined as usual, where the idea is that the element $s_i(j)$ is assigned to the variable v_{ij}. If φ is a sentence of $\mathscr{L}_{\infty\omega}(K)$ and X is a class of structures, we say φ is valid in X if each structure of X is a model of φ. φ is valid if it is valid in the class of all K-structures. We note that a 0-ary relation symbol P is true in a structure $\langle M, \rho\rangle$ if $\rho(P)=0$.

We shall often confine ourselves to formulas involving only the variables v_i, $i\in\omega$. Such a formula is an n-formula if its free variables are among the v_i for $i\in n$. In this case we may write $\mathfrak{M}\models\varphi[m_0,\ldots,m_{n-1}]$ for $\mathfrak{M}\models\varphi[s_0]$ where $s_0(i)=m_i$, $0\leqslant i\leqslant n$.

Henceforth ε is a fixed binary relation symbol and U is a fixed unary relation symbol. We shall be primarily concerned below with the alphabets $L(\varepsilon, \mathrm{U})=\{=, \varepsilon, \mathrm{U}\}$ and $L(\varepsilon, \mathrm{U}, P_1,\ldots,P_n)=L(\varepsilon,\mathrm{U})\cup\{P_1,\ldots,P_n\}$ where the P_i's are unary relation symbols (in $\mathrm{HF}\cap V_0$).

We should also remark that many of the logical symbols of the metalanguage (e.g., $\forall, \exists, \wedge, \vee, \neg$) will also be used in names for formulas of $L(\varepsilon,\mathrm{U})$. Which way such a symbol is being used should be clear from the context, as the variables of $L(\varepsilon,\mathrm{U})$ are named by u, v. Furthermore, we use ε, U rather than $\in$, U in $L(\varepsilon,\mathrm{U})$.

A formula of $L(\varepsilon,\mathrm{U},P_1,\ldots,P_n)$ is Δ_0 if it is built up from atomic formulas by means of $\wedge$, $\vee$, $\neg$ and $\forall v_i\varepsilon v_j$ where $\forall v_i\varepsilon v_j$ is $\forall v_i(v_i\varepsilon_j\to\theta)$. (Similarly $\exists v_i\varepsilon v_j\theta$ is $\exists v_i(v_i\varepsilon v_j\wedge\theta)$.) We write $\exists u, v\varepsilon w\theta$ for $\exists u\varepsilon w\exists v\varepsilon w\theta$. Similar notations will appear and should be interpreted analogously. A formula of the form $\exists v_i\theta$ where θ is Δ_0, is called a Σ_1 formula. A formula is Σ if it is built up from Δ_0 formulas by means of $\wedge$, $\vee$, $\forall v_i\varepsilon v_j$, and $\exists v_i$, for $i,j\in\omega$.

A structure $\mathfrak{N}=\langle N, \varepsilon^{\mathfrak{N}}, \mathrm{U}^{\mathfrak{N}},\ldots\rangle$ is an endextension of a K-structure $\mathfrak{M}=\langle M, \varepsilon^{\mathfrak{M}}, \mathrm{U}^{\mathfrak{M}},\ldots\rangle$ if $\mathfrak{M}\subseteq\mathfrak{N}\upharpoonright K$ and, whenever $m\in M$, $n\in N$ and $n\varepsilon^{\mathfrak{N}}m$, then $n\in M$ In this case if φ is a Δ_0 n-formula and $a\in M^n$, $\mathfrak{M}\models\varphi[s]$ if and only if $\mathfrak{N}\models\varphi[s]$. This can be established by a simple induction on Δ_0 formulas.

A relation R on an $L(\varepsilon,\mathrm{U},P_1,\ldots,P_n)$-structure is $\Delta_0(\Sigma_1,\Sigma)$ if it is definable, with parameters, on the structure by a formula which is Δ_0 (resp. Σ_1, Σ). A relation R is $\Delta(\Delta_1)$ on the structure if both R and its complement in the structure are $\Sigma(\Sigma_1)$.

KPU $(P_1,\ldots,P_n)$ is the set of the universal closures of the following formulas of $L(\varepsilon,\mathrm{U},P_1,\ldots,P_n)$:

(0) $\exists v_0(\neg\mathrm{U}v_0\wedge\forall v_1\varepsilon v_0(\neg v_1=v_1))$ (Empty Set);

(1) $\exists v_2(\text{“}v_2=\{v_0,v_1\}\text{”})$ (Pairing);

(2) $(\forall v_1\varepsilon v_0)(\neg\mathrm{U}v_1)\to(\exists v_1)(\text{“}v_1=\bigcup v_0\text{”})$ (Union);

(3) $\exists v_1\forall v_2(v_2\varepsilon v_1\leftrightarrow(v_2\varepsilon v_0\wedge\varphi))$ where φ is Δ_0, and v_1 and v_2 are not free in φ (Δ_0-Separation);

(4) $\forall v_1\varepsilon v_0\exists v_2\varphi\to\exists v_3\forall v_1\varepsilon v_0\exists v_2\varepsilon v_3\varphi$, where φ is Δ_0 and v_3 is not free in φ (Δ_0-Collection);

(5) $v_0\varepsilon v_1\to\neg\mathrm{U}v_1$; and

(6) $\exists v_0\varphi(v_0)\to\exists v_0(\varphi(v_0)\wedge\forall v_1\varepsilon v_0(\neg\varphi(v_1)))$ where φ is an arbitrary formula of $L(\varepsilon,\mathrm{U},P_1,\ldots,P_n)$ and v_1 does not occur in φ.

The reader should recall (or discover) (i) that (4) is a theorem of $\mathrm{KPU}(P_1, \ldots, P_n)$ for all φ which are Σ_1, and (ii) that for every Σ formula θ of $L(\varepsilon, \mathrm{U}, P_1, \ldots, P_n)$, there is a Σ_1 formula θ_1 of $L(\varepsilon, \mathrm{U}, P_1, \ldots, P_n)$, such that $(\theta \leftrightarrow \theta_1)$ is a theorem of $\mathrm{KPU}(P_1, \ldots, P_n)$.

The case $n = 0$ is permitted above. This means simply that there are no P_i's and $\mathrm{KPU} = \mathrm{KPU}(0)$ is formulated purely in terms of ε, U, $=$. If A is a transitive set, we regard the structure $\mathbf{A} = \langle A, \varepsilon \cap A^2, U(A)\rangle$ as an $L(\varepsilon, \mathrm{U})$-structure in the obvious way.

Let $\mathscr{A} = \langle \mathbf{A}, R_1, \ldots, R_n\rangle$ be an $L(\varepsilon, \mathrm{U}, P_1, \ldots, P_n)$-structure. We say that $\mathscr{A}$ is admissible, or A is $R_1, \ldots, R_n$-admissible if $\mathscr{A}$ is a model of $\mathrm{KPU}(P_1, \ldots, P_n)$, that is, if

(0) $\emptyset \in A$

(1) $x, y \in A \to \{x, y\} \in A$,

(2) whenever $b \in A$ is a set of sets, $\bigcup b \in A$,

(3) whenever $X \subseteq A$ is Δ_0 on $\mathscr{A}$ and $a \in A$, $X \cap a \in A$ (Δ_0-separation), and

(4) whenever $X \subseteq A^2$ is Δ_0 on $\mathscr{A}$, $a \in A$, and $\forall x \in a\, \exists y \in A \quad Xxy$, then there is $b \in A$ such that $\forall x \in a\, \exists y \in b \quad Xxy$ (Δ_0-Collection).

We say A is admissible if $\mathbf{A}$ is a model of KPU. If A is admissible, an element of A is said to be A-finite. The ordinal of A is the least ordinal not in A.

A binary relation R on a set X is well-founded if R has no descending chain, i.e., if there is no $f : \omega \to X$ such that $Rf(i+1)f(i)$ for all $i \in \omega$. If $\mathfrak{N} = \langle N, E, U'\rangle$ is an $L(\varepsilon, \mathrm{U})$-structure, then $\mathrm{WF}(\mathfrak{N})$, the well-founded part of $\mathfrak{N}$, is the set of $n \in N$ such that there is no $f : \omega \to N$ such that $f(0) = n$ and $f(i+1)\, Ef(i)$ for all $i \in \omega$. $E \restriction \mathrm{WF}(\mathfrak{N})^2$ is then well-founded. Further, $\mathfrak{N}$ is an endextension of

$$\mathfrak{N}^{\mathrm{WF}} = \langle \mathrm{WF}(\mathfrak{N}), E \cap \mathrm{WF}(\mathfrak{N})^2, U' \cap \mathrm{WF}(\mathfrak{N})\rangle.$$

if $\mathfrak{N}$ is a model of KPU, $U' \subseteq \mathrm{WF}(N)$. Suppose that U' is a set of urelements. Then, using axiom (2), it can be seen that $\mathfrak{N}^{\mathrm{WF}}$ is isomorphic to $A_{\mathfrak{N}}$ for a transitive set $A_{\mathfrak{N}} \in V_{U'}$ by a unique isomorphism fixing the elements of U'. We say $\mathfrak{N}$ is normal if $\mathrm{U}^{\mathfrak{N}}$ is a set of urelements and this isomorphism is the identity, that is $\mathfrak{N}^{\mathrm{WF}} = \mathbf{A}_{\mathfrak{N}}$. Every model of KPU is isomorphic to a normal model.

We shall need the Truncation Lemma of Barwise [1, 2, 3]: If $\mathfrak{N}$ is a normal model of KPU, $\mathfrak{N}^{\mathrm{WF}}$ is admissible. (To show that $\mathfrak{N}^{\mathrm{WF}}$ satisfies Δ_0-Collection, an application of Foundation in $\mathfrak{N}$ is necessary.)

Theorem 1.1. *There is a Σ_1-formula $\Pr(v_0)$ such that, whenever A is countable admissible, and $\varphi \in A$, $\mathbf{A} \models \Pr[\varphi]$ if and only if $\varphi \in \mathscr{L}_{\infty\omega}$ and φ is valid.*

Proof. $\Pr(v_0)$ says essentially that "v_0 has a proof" in a certain (infinitary) formal system. The Δ_0-collection axiom is used in treating infinitary disjunctions in the completeness proof for the system. Details may be found in [3]. □

We shall write "$\vdash\varphi$" to mean that $\mathbf{V}\models \Pr[\varphi]$ and use also the notation "$Y\vdash\varphi$" for $\vdash \bigwedge Y \to \varphi$.

The Barwise Compactness Theorem [1] states that a collection of sentences, Σ on a countable admissible set A, every A-finite subset of which has a model; has a model. To prove a generalisation of this theorem in the next section, we shall require the following theorem of Karp, proved in [10].

Theorem 1.2. *Let A be countable and PR closed and let X be a set of sentences of A. Suppose that for every sentence of the form $\bigwedge\Phi$ in A, whenever $\forall\varphi\in\Phi\exists x\in A$ $(x\subseteq X\wedge x\vdash\varphi)$, there is $x'\in A$, such that $x'\subseteq X$ and $x'\vdash\bigwedge\Phi$. Suppose also that every A-finite subset of X has a model. Then X has a model.*

Let $\mathcal{F}$ be a countable subset of $\mathcal{L}_{\infty\omega}(K)$ closed undertaking of subformulas. We call such a set of countable fragment of $\mathcal{L}_{\infty\omega}(K)$. A formula is $\bigvee\exists$ over $\mathcal{F}$ if it is of the form

$$\bigvee_{n\in\omega} \exists v_0\cdots v_n(\varphi_1^n\wedge\cdots\wedge\varphi_n^n)$$

where each φ_i^j is in $\mathcal{F}$. A class of structures is $\forall\bigvee\exists$ over $\mathcal{F}$ if it is the class of models of a set of sentences of the form $\forall v\psi(v)$, where ψ is $\bigvee\exists$ over $\mathcal{F}$ and v is some sequence of variables.

We shall need the following theorem (cf. [11]).

Theorem 1.3. *Let X be a nonempty class of K-structures which is $\forall\bigvee\exists$ over $\mathcal{F}$. For each $n\in\omega$, let γ_n be the formula $\forall v_0\cdots v_{m_n}\tau_n(v_0,\ldots,v_{m_n})$ where $m_n\in\omega$ and τ_n is $\bigvee\exists$ over $\mathcal{F}$. Suppose that whenever $\eta(v_0\cdots v_k)$ is $\bigvee\exists$ over $\mathcal{F}$ and $\exists v_0\cdots v_k\eta$ has a model in X, $\exists v_0\cdots v_l(\eta\wedge\tau_n)$ has a model in X, where $l=\max(k,m_n)$. Then X contains a model of $\bigwedge\{\gamma_n : n\in\omega\}$.*

2. $A(\mathfrak{M})$

We assume throughout this section that A is a transitive set closed under PR functions. We let α be the ordinal of A. Given a structure $\mathfrak{M}=\langle M,p\rangle$, where M is a set of urelements disjoint from A, we shall define a set $A(\mathfrak{M})$ including A and characterise it in a number of ways. This set is almost the primitive recursive closure of $A\cup\{\mathfrak{M}\}$, and is exactly that when $\mathfrak{M}$ is a K-structure for some $K\in A$.

The development below is modelled somewhat on unramified forcing constructions (see, for example, Shoenfield [17]). Elements of the "inner model" A will serve as names for the elements of the "outer model" $A(\mathfrak{M})$ and the structure of $A(\mathfrak{M})$ is reduced to certain relations between the names and $\mathfrak{M}$. However, some differences from forcing are necessary, due to our interest in "nongeneric" $\mathfrak{M}$. More drastic modifications are necessitated by the need to name elements of M. For this we use "variables". The exact meaning of this can be seen in the development following.

First we shall introduce some useful notation. If φ is a formula, $\varphi(i_0 \cdots i_n)$ is the formula obtained by (simultaneously) substituting in φ the variable $v_{i_k j}$ for v_{kj} for $0 \leq k \leq n$ and all j. Thus, for example, if $s_0, s_1 \in M^n$ and φ is a formula containing free only the variables v_{ij} for $i = 0, 1$, and $0 \leq j \leq n$, then $\mathfrak{M} \models \varphi[s_0, s_1]$ if and only if $\mathfrak{M} \models \varphi(1, 0)[s_1, s_0]$.

We need furthermore a notion like $\mathfrak{M} \models \varphi$ even when $\mathfrak{M}$ does not interpret all the relations and function symbols of φ. We arrange this as follows: given $\varphi \in \mathscr{L}_{\infty\omega}$ and $\mathfrak{M}$ a K-structure, let $H_K(\varphi)$ be obtained by replacing all occurrences inside φ of an atomic formula $Rt_1 \cdots t_n$ involving any symbol not in K by $\mathbb{T}$; then we say that $\mathfrak{M} \models \varphi[s_0, \ldots, s_n]$ if $\mathfrak{M} \models H_K(\varphi)[s_0, \ldots, s_n]$, for all sequences s_i from M. If $\mathfrak{M}$ interprets all the symbols of φ, then $H_K(\varphi) = \varphi$ and we obtain the old notion $\models$. Note also that the function $G(K, \varphi) = H_K(\varphi)$ is PR.

Assume henceforth that A is primitive recursively closed, and $\mathfrak{M} = \langle M, \rho \rangle$ is a K-structure for some $K \subseteq A$, with universe M a set of urelements disjoint from A.

We shall need the following function L:

$$L(x) = \begin{cases} n+1 & \text{if } x = n \in \omega, \\ k & \text{if (i) } x \text{ is a function, (ii) for all } y \in \operatorname{dom} x \\ & x(y) \in \mathscr{L}_{\infty\omega}, \text{ (iii) the free 0-variables of } x(y) \\ & \text{are among } v_{00}, \ldots, v_{0(k-1)}, \text{ and (iv) } k \text{ is the} \\ & \text{least natural number with this property,} \\ 0 & \text{otherwise.} \end{cases}$$

Then L is primitive recursive.

There is a primitive recursive predicate Nx ("x is a name") which satisfies: $Nx \leftrightarrow x \in U \vee x \in \omega \vee (\exists k \in \omega)$ (x is a non-empty function $\wedge$ dom $x \subseteq N$ $\wedge$ $(\forall y \in \operatorname{dom} x)(x(y) \in \mathscr{L}_{\infty\omega}$ $\wedge$ the free variables of $x(y)$ are among $\{v_{00}, \ldots, v_{0(k-1)}, v_{10}, \ldots, v_{1(L(y)-1)}\}$)).

Hence x is a name if it is an urelement, a natural number, or a function on names each of whose values is a formula in a set of free 0-variables depending on x alone.

Roughly, urelements will denote themselves, natural numbers will denote "new" urelements, and each name which is a function f denotes the set consisting of those x denoted by a name y in the domain of f for which $f(y)$ is true in $\mathfrak{M}$. The major inaccuracy of this description is that it does not state the role of variables. We do this now with a precise definition.

By primitive recursion we define a function D as follows, such that for all $\mathfrak{M} = \langle M, \rho \rangle$, $x \in N$, $D(\mathfrak{M}, x) : M^{L(x)} \to V$:

$$D(\mathfrak{M}, x) = 0 \quad \text{if} \quad x \notin N$$

$$D(\mathfrak{M}, x) = \{\langle 0, x \rangle\} \quad \text{if} \quad x \in U$$

$$D(\mathfrak{M}, n) = \{\langle s, s(n) \rangle : s \in M^{n+1}\} \quad \text{if} \quad n \in \omega$$

$$D(\mathfrak{M}, f) = \{\langle s, \{D(M, x)(t) : x \in \operatorname{dom} f \wedge t \in M^{L(x)} \wedge \mathfrak{M} \models f(x)[s, t]\}\rangle : s \in M^{L(f)}\} \quad \text{otherwise}$$

D is the denotation function. For fixed $\mathfrak{M}$, it assigns a denotation to pairs $\langle x, s\rangle$ where $x \in N$ and $s \in M^{L(x)}$. Let us consider some cases of this.

First, suppose $M=0$ (a condition we emphatically permit). Then $M^{L(x)}=0$ unless $L(x)=0$. So $D(\mathfrak{M}, x)$ will be empty unless $L(x)=0$.

The more complex case involves nonempty M. If x is an urelement, then $L(x)=0$, and $D(\mathfrak{M}, x)(0)$ is simply x. So the pair $\langle x, 0\rangle$ denotes x. Consider now $n \in \omega$. If $s \in M^{n+1}$, $\langle n, s\rangle$ denotes $s(n) \in M$. Clearly each element of M is $D(\mathfrak{M}, 0)(s)$ for some $s \in M^1$.

Now consider the case of a function f in N for which $L(f)=k$. Then if $s \in M^k$, the denotation of $\langle f, s\rangle$ is simply $\{z$: there is $y \in \operatorname{dom} f$, $t \in M^{L(y)}$ such that $\mathfrak{M} \models f(y)[s, t]$ and z is the denotation of $\langle y, t\rangle\}$. For example, suppose $x_1 = D(\mathfrak{M}, x_1')(s_1)$, $x_2 = D(\mathfrak{M}, x_2')(s_2)$. Let $L(x_1)=k$, $L(x_2)=l$. Define f as follows. $\operatorname{dom} f=\{x_1', x_2'\}$. $f(x_1')$ is the formula

$$v_{10}=v_{00} \wedge v_{11}=v_{01} \wedge \cdots \wedge v_{1(k-1)}=v_{0(k-1)}.$$

$f(x_2')$ is the formula

$$v_{10}=v_{0k} \wedge v_{11}=v_{0(k+1)} \wedge \cdots \wedge v_{1(l-1)}=v_{0(k+l-1)}.$$

It is not difficult to verify that $D(\mathfrak{M}, f)(s_1 \frown s_2)$ is $\{x_1, x_2\}$.

As another example, define f as follows: $\operatorname{dom} f=\{0\}$; $f(0)$ is $v_{10}=v_{10}$. Then $L(f)=0$ and

$$\begin{aligned} D(\mathfrak{M}, f)(0) &= \{D(\mathfrak{M}, 0)(s): s \in M^1 \wedge \mathfrak{M} \models (v_{10}=v_{10})[0, s]\} \\ &= \{s(0): s \in M^1\} = M. \end{aligned}$$

Definition 2.1. For B a transitive set, let $B(\mathfrak{M}) = \bigcup \{\text{Range } D(\mathfrak{M}, x): x \in B\}$.

We are interested in $A(\mathfrak{M})$, where A is the transitive primitive recursively closed set fixed above.

Proposition 2.1. (i) $A \subseteq A(\mathfrak{M})$. $M \in A(\mathfrak{M})$. *$A(\mathfrak{M})$ is transitive and* $U(A(\mathfrak{M})) = U(A) \cup M$. *The ordinal of* $A(\mathfrak{M})$ *is* α.

(ii) *Suppose* $K_0 \in A$ *and* $K_0 \subseteq K$. *Then* $\mathfrak{M} \upharpoonright K_0 \in A(\mathfrak{M})$.

Proof. (i) We define a primitive recursive function $\check{x}$ as follows:

$$\check{x} = x \quad \text{if} \quad x \in U,$$

$$\check{a} = \{\langle \check{b}, \boldsymbol{T}\rangle : b \in a\} \quad \text{if } a \text{ is a set.}$$

Then $\check{x} \in N$ for all x, and for any $\mathfrak{M}$, $D(M, \check{x})(0) = x$. If $x \in A$, $\check{x} \in A$ since A is PR-closed, so $A \subseteq A(\mathfrak{M})$.

As noted earlier $M = D(\mathfrak{M}, a)(0)$ where $a = \langle 0, v_{10}=v_{10}\rangle$. Since clearly $a \in A$, $M \in A(\mathfrak{M})$.

Urelements appear as $D(\mathfrak{M}, x)(s)$ only if $x \in U$ or $x \in \omega$. It easily follows that $U(A(\mathfrak{M})) = U(A) \cup M$.

An easy induction on $x \in N$ shows that $\operatorname{rk}(D(\mathfrak{M}, x)(s)) \leqslant \operatorname{rk}(x)$. Hence the ordinal of $A(\mathfrak{M})$ is not greater than the ordinal of A. But $A \subseteq A(M)$, so their ordinals are equal.

(ii) It is easily proven using the remarks above that there are A_n, for $n \in \omega$, definable primitive recursively, such that $L(A_n) = n$, $D(\mathfrak{M}, A_n)(s) = \langle s(0), \ldots, s(n)\rangle$ for all $s \in M^n$.

If P is an n-ary relation symbol, let

$$P' = \langle A_{n-1}, Pv_{10} \cdots v_{1(n-1)} \rangle.$$

Then $L(P') = n$ and $D(\mathfrak{M}, P')(0) = P^{\mathfrak{M}}$, if $P \in K$, and M^n otherwise. From this remark and the construction above for order pairs it is easy to find a name K_0^*, for each $K_0 \in A$ such that $K_0 \subseteq K$, for which $D(\mathfrak{M}, K_0^*)(0) = \mathfrak{M} \upharpoonright K_0$. $\square$

We now begin a more extensive analysis of $A(\mathfrak{M})$. Our next proposition shows that equality of $D(\mathfrak{M}, x)(s)$ and $D(\mathfrak{M}, y)(t)$ is equivalent to the satisfaction in $\mathfrak{M}$ by $s^\frown t$ of a certain formula.

Proposition 2.2. *There is an $F_1 : A^2 \to A$ which is primitive recursive and satisfies:*

(i) *for all $x, y \in N$, $F_1(x, y)$ is a formula ψ of $\mathscr{L}_{\infty\omega}$ with free variables v_{0j}, v_{1k}, $j \in L(x)$, $k \in L(y)$, and*

(ii) *for all $\mathfrak{M}$, $s \in M^{L(x)}$, $t \in M^{L(y)}$*

$$D(\mathfrak{M}, x)(s) = D(\mathfrak{M}, y)(t) \leftrightarrow \mathfrak{M} \models \psi[s, t].$$

Proof. We define $F_1(x, y)$ by simultaneous recursion on x and y from PR functions. By results in [9], F_1 will be PR.

If $\neg x \in N$ or $\neg y \in N$, let $F_1(x, y) = 0$, say.
If $x \in \omega \cup U$ and $y \in N$, or vice versa, let

$$F_1(x, y) = \begin{cases} \mathbf{T} & \text{if } x \in U,\ y \in U, \text{ and } x = y, \\ v_{0x} = v_{1y} & \text{if } x \in \omega,\ y \in \omega, \\ \mathbf{F} & \text{otherwise.} \end{cases}$$

Otherwise, let $F_1(x, y)$ be the conjunction of

$$(1) \qquad \bigwedge_{z \in \operatorname{dom} x} \forall v^2 [x(z)(02) \to \bigvee_{w \in \operatorname{dom} y} \exists v^3 (y(w)(13) \wedge F_1(z, w)(23))]$$

and

$$(2) \qquad \bigwedge_{w \in \operatorname{dom} y} \forall v^3 [y(w)(03) \to \bigvee_{z \in \operatorname{dom} x} \exists v^2 (x(z)(12) \wedge F_1(z, w)(23))],$$

where $v^2(v^3)$ is a sequence of the variables v_{2i}, $i \in L(z)$ (resp. v_{3i}, $i \in L(w)$).

It is clear that F_1 so defined satisfies condition (i) of the theorem so we set out to verify (ii). This is clear for all y when x is an urelement or a natural number.

Suppose now that $x \in N$ and (ii) is satisfied for all z of rank less than rk (x) and all y. We shall verify (ii) for all y. Fixing y, we may assume $y \in U \cup \omega$ (otherwise

(ii) is clear) and hence that $F_1(x, y)$ is the conjunction of (1) and (2) above. Let φ be the formula in (1).

We shall prove that

$$D(\mathfrak{M}, x)(s) \subseteq D(\mathfrak{M}, y)(t) \leftrightarrow \mathfrak{M} \models \varphi[s, t].$$

We then leave it to the reader to verify the same equivalence for the converse inclusion and the formula in (2), thereby completing the proof.

Suppose first that $D(\mathfrak{M}, x)(s) \subseteq D(\mathfrak{M}, y)(t)$. We show that $\mathfrak{M} \models \varphi[s, t]$. Let $z \in \operatorname{dom} x$, $s' \in M^{L(z)}$ and suppose that

(3) $\mathfrak{M} \models (x(z)(02))[s, t, s']$.

That is, decoding the notation,

(4) $\mathfrak{M} \models x(z)[s, s']$.

By the definition of $D(\mathfrak{M}, x)$, $D(\mathfrak{M}, z)(s') \in D(\mathfrak{M}, x)(s)$. Since $D(\mathfrak{M}, x)(s) \subseteq D(\mathfrak{M}, y)(t)$, $D(\mathfrak{M}, z)(s') \in D(\mathfrak{M}, y)(t)$. By definition of $D(\mathfrak{M}, y)$, there is $w \in \operatorname{dom} y$ and $t' \in M^{L(y)}$ such that

(5) $D(\mathfrak{M}, z)(s') = D(\mathfrak{M}, w)(t')$, and

(6) $\mathfrak{M} \models y(w)[t, t']$.

But then, using notation again,

(7) $\mathfrak{M} \models (y(w)(13))[s, t, s', t']$

and, by the induction hypothesis and (5)

(8) $\mathfrak{M} \models F_1(z, w)[s', t']$.

But then t' witnesses, as necessary, the existential quantifier $\exists v^3$ in (1). Hence, $\mathfrak{M} \models \varphi[s, t]$.

Suppose conversely that $\mathfrak{M} \models \varphi[s, t]$ and let $a \in D(\mathfrak{M}, x)(s)$. There is $z \in \operatorname{dom} x$, $s' \in M^{L(z)}$ such that $a = D(\mathfrak{M}, z)(s')$ and

(9) $M \models x(z)[s, s']$.

Hence by the notation

(10) $\mathfrak{M} \models (x(z)(02))[s, t, s']$.

But then, since $\mathfrak{M} \models \varphi[s, t]$, there is $w \in \operatorname{dom} y$ and $t' \in M^{L(w)}$ such that

(11) $\mathfrak{M} \models (y(w)(0, 3) \wedge F_1(z, w)(23))[s, t, s', t']$.

But then

(12) $\mathfrak{M} \models y(w)[t, t']$, and

(13) $\mathfrak{M} \models F_1(z, w)[s', t']$.

By (13) and the induction hypothesis, $D(\mathfrak{M}, w)(t') = D(\mathfrak{M}, z)(s') = a$, and by (12), $D(\mathfrak{M}, w)(t') \in D(\mathfrak{M}, y)(t)$. Hence $a \in D(\mathfrak{M}, y)(t)$ and so $D(\mathfrak{M}, x)(s) \subseteq D(\mathfrak{M}, y)(t)$, as required, completing the proof. □

Corollary 2.3. *There is a primitive recursive function F_2 satisfying:*

(1) *for all $x, y \in N$, $F_2(x, y)$ is a formula θ of $\mathcal{L}_{\infty\omega}$ with free variables among v_{0i} and v_{1j} for $i \in L(x)$ and $j \in L(y)$*

(2) *for all $x, y \in N$, and all $\mathfrak{M}s \in M^{L(x)}$, $t \in M^{L(y)}$, $D(\mathfrak{M}, x)(s) \in D(\mathfrak{M}, y)(t)$ if and only if $\mathfrak{M} \models F_2(x, y)[s, t]$.*

Proof. Let

$$F_2(x, y) = \bigvee_{w \in \operatorname{dom} y} \exists v^2[y(w)(12) \wedge F_1(x, w)(02)]$$

where v^2 is a sequence of the variables v_{2j}, for $j \in L(w)$. That this F_2 works follows easily from Proposition 2.2. □

Proposition 2.4. *$A(\mathfrak{M})$ is primitive recursively closed.*

Proof. We shall define for each PR function F a PR function G_F such that

$$F(D(\mathfrak{M}, x_1)(s_1), \ldots, D(\mathfrak{M}, x_n)(s_n)) \\ = D(\mathfrak{M}, G_F(x_1, \ldots, x_n)) \quad (s_1{}^\frown \cdots {}^\frown s_n) \quad (1)$$

for all $x_i \in N$ and $s_i \in M^{L(x_i)}$ for $1 \leq i \leq n$. This clearly implies the conclusion of the theorem.

The definition is by recursion on PR functions.

Suppose $F(y_1, \ldots, y_n) = y_i$ for all $y_1, \ldots, y_n$. For simplicity we consider the case $i = 1$. (The other cases will differ only by a substitution of variables.) But then $G_F(x_1, \ldots, x_n) = x_1$ satisfies (1). If F is the 0-function, the constant function with value 0 is PR and satisfies (1).

The pairing operation was handled in the remarks preceding Definition 2.1. We consider now

$$F(x) = \begin{cases} \bigcup x & \text{if } x \text{ is a set of sets,} \\ 0 & \text{otherwise.} \end{cases}$$

We may define G_F as follows: If $x \notin N$ or $x \in U$, let $G_F(x) = 0$; if $x \in \omega$, let $G_F(x) = \{(0, \neg v_{0x} = v_{0x})\}$. Otherwise let $T = \bigcup \{\operatorname{dom} y : y \in \operatorname{dom} x\}$; if $T \neq 0$, let $\operatorname{dom} G_F = T$ and for $u \in T$, let

$$G_F(x)(u) = \left[\bigvee_{\substack{u \in \operatorname{dom} y \\ y \in \operatorname{dom} x}} \exists v^2(y(u)(2) \wedge x(y)(02)) \right] \\ \wedge \left[\neg \bigvee_{y \in \operatorname{dom} x \cap (\omega \cap u)} \exists v^2(x(y)(02)) \right]$$

where v^2 is a sequence of the variables v_{2i}, $i \in L(y)$; if $T = 0$ let $G_F(x) = 0$. It is fairly easily verified that G_F satisfies (1).

We leave the induction step for composition to the reader and verify the primitive recursion step. For simplicity we assume that we are considering the recursion

$$F(x) = H(\bigcup \{F(u) : u \in x\}, x)$$

and we have G_H such that

$$H(D(\mathfrak{M}, x)(s), D(\mathfrak{M}, y)(t)) = D(\mathfrak{M}, G_H(x, y))(s \frown t)$$

for all $x, y \in N$, $s \in M^{L(x)}$, and $t \in M^{L(y)}$. Let S be the function defined (as G_L) in the previous paragraph.

We define G_F by primitive recursion as follows:

$$G_F(x) = J_1(G_H(S(J_2(x)), x)),$$

where

$$\operatorname{dom} J_2(x) = \{G_F(u) : u \in \operatorname{dom} x\}$$

and

$$J_2(x)(y) = \mathbb{W}\{x(u) : u \in \operatorname{dom} x \wedge G_F(u) = y\}$$

and $J_1(G_H(H_2(x), X))$ is the result of replacing v_{0i} in $G_H(J_2(x), x)(u)$ by $v_{0(i-L(x))}$ for all i satisfying $L(x) \leqslant i \leqslant 2L(x)$, for all $u \in \operatorname{dom} G_H(J_2(x), x)$.

Let us show by induction on x that G_F satisfies (15). If

$$F(D(\mathfrak{M}, u)(t)) = D(\mathfrak{M}, G_F(u))(t)$$

for all $u \in N$ of rank less than x and all $t \in M^{L(u)}$, then

$$\begin{aligned} D(\mathfrak{M}, J_2(x))(s) &= \{D(\mathfrak{M}, G_F(u))(t) : u \in \operatorname{dom} x \wedge t \in M^{L(u)} \wedge \mathfrak{M} \models x(u)[s, t]\} \\ &= \{F(D(\mathfrak{M}, u)(t)) : u \in \operatorname{dom} x \wedge t \in M^{L(u)} \wedge \mathfrak{M} \models x(u)[s, t]\}, \\ &\qquad \text{by the induction hypothesis} \\ &= \{F(y) : y \in D(\mathfrak{M}, x)(s)\}. \end{aligned}$$

Therefore if $s \in M^{L(x)}$,

$$\begin{aligned} D(\mathfrak{M}, G_F(x))(s) &= D(\mathfrak{M}, J_1(G_H(S(J_2(x)), x)))(s) \\ &\qquad \text{by definition of } G_F \\ &= D(\mathfrak{M}, G_H(J_2(x), x))(s \frown s) \\ &\qquad \text{by definition of } J_1 \\ &= H(D(\mathfrak{M}, S(J_2(x)))(s), D(\mathfrak{M}, x)(s)) \\ &\qquad \text{by induction hypothesis on } G_H \\ &= H(\bigcup \{F(y) : y \in D(\mathfrak{M}, x)(s)\}, D(\mathfrak{M}, x)(s)) \\ &\qquad \text{by the preceding computation} \\ &= F(D(\mathfrak{M}, x)(s)), \end{aligned}$$

as required. This completes the proof. □

Corollary 2.5. *Suppose A is PR (χ_K)-closed and $\mathfrak{M}$ is a K-structure. Then $A(\mathfrak{M})$ is the* PR *closure of* $A \cup \{\mathfrak{M} \upharpoonright K_0 : K_0 \in A, K_0 \subseteq K\}$.

Proof. $A(\mathfrak{M})$ Is PR closed. We need only show that $A(\mathfrak{M}) \subseteq C$ where C is the PR-closure of $A \cup \{\mathfrak{M} \upharpoonright K_0 : K_0 \in A, K_0 \subseteq K\}$.

Let $x \in N$, and let $K_0 = K \cap \mathrm{TC}(x)$. Then $K_0 \in A$ since A is closed under PR (K) functions. Now if $\varphi \in \mathscr{L}_{\infty\omega} \cap \mathrm{TC}(x)$ is an n-formula and $s \in M^n$ for some n, $\mathfrak{M} \models \varphi[s] \leftrightarrow \mathfrak{M} \upharpoonright K_0 \models \varphi[s]$. It follows easily that if $y \in \mathrm{TC}(x)$ and $s \in M^{L(y)}$, $D(\mathfrak{M}, y)(s) = D(\mathfrak{M} \upharpoonright K_0, y)(s)$. But from this it is clear that $A(\mathfrak{M}) \subseteq C$ as required, since D is a primitive recursive function of its two arguments. □

One simple consequence of the corollary is that $A(\mathfrak{M})$ is just

$$\bigcup \{L_\alpha(\{a, \mathfrak{M} \upharpoonright K_0\}) : K_0 \subseteq K, K_0 \in A, \text{ and } a \in A\},$$

when A is closed under PR (χ_K) functions.

For each element of $L_\alpha(\{a, \mathfrak{M} \upharpoonright K_0\})$ is an element of $L_\beta(\{a, \mathfrak{M} \upharpoonright K_0\})$ for some $\beta \in A$ and remarks in Section 1 show that

$$G(x, y, z) = \begin{cases} L_x(\{y, z\}) & \text{if } x \in \mathrm{On}, \\ 0 & \text{otherwise.} \end{cases}$$

is PR.

The converse inclusion can be obtained as follows: for $x \in A \cap N$, let $K_0 \in A$ be $K \cap \mathrm{TC}(x)$. Then it is easy to show that there is a PR function G satisfying

$$D(\mathfrak{M} \upharpoonright K_0, x) \in L_{G(K_0, x)}(\{\mathfrak{M} \upharpoonright K_0, x\}).$$

As an alternative, one may apply the Stability Theorem of [9].

We of course might have directly defined $A(\mathfrak{M})$ to satisfy the above equation. Several technical advantages of the method used governed our choice.

The following theorem continues the reductions above. It shall be very useful in later chapters, and is used in one major application later in this section.

Theorem 2.6. *There is a primitive recursive function F of two variables such that if $\varphi(v_0, \ldots, v_n)$ is a Δ_0-formula of $L(\varepsilon, U)$ with at most the displayed variables free, and $x_i \in N$ for $0 \leq i \leq n$, then*

(1) *$F(\varphi, \langle x_0, \ldots, x_n \rangle)$ is a formula ψ of $\mathscr{L}_{\infty\omega}$ with free variables v_{ij} for $0 \leq i \leq n$, $j \in L(x_i)$; and*

(2) *for all M and all $s_i \in M^{L(x_i)}$, $0 \leq i \leq n$,*

$$\varphi(D(\mathfrak{M}, x_0)(s_0), \ldots, D(\mathfrak{M}, x_n)(s_n)) \leftrightarrow \mathfrak{M} \models F(\varphi, \langle x_0, \ldots, x_n \rangle)[s_0, \ldots, s_n].$$

Proof. This can be easily proven by induction on φ. Note first that Proposition 2.2 and Corollary 2.3 provide us with $F(\varphi, y)$ whenever φ is of the form $v_i = v_j$ or $v_i \varepsilon v_j$.

The induction steps for $\neg$ and $\wedge$ are trivial. If φ is $\forall v_i \varepsilon v_j \psi$ then let

$$F(\varphi, \langle x_1, \ldots, x_n \rangle)$$
$$= \bigwedge\!\!\!\bigwedge_{y \in \operatorname{dom} x_j} \forall v^i [x_j(y)(i) \to F(\psi, \langle x_0, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n \rangle)]$$

where v^i is a sequence of the variables v_{ij}, $j \in L(x_i)$.

It is easy to verify (2) above, and this is left to the reader. □

The reduction of Proposition 2.6 can easily be extended to a class of formulas involving infinitary conjunctions and disjunctions as well as negation and bounded quantifications.

It cannot, however, admit extension to allow formulas with an extra relation symbol to be interpreted as the diagram of $\mathfrak{M}$. The author does not know exactly what the best possible extension of 2.6 is.

We are now in a position to give a striking characterisation of $A(M) \cap V_{U(A)}$.

Definition 2.2. By induction on β, we define for all n and $s \in M^n$, an n-formula $\delta^{\mathfrak{M}}_{\beta,s} \in \mathcal{L}_{\infty\omega}(K)$, as follows:

(i) $\delta^{\mathfrak{M}}_{0,s}$ is the (infinitary) conjunction of n-formulas ψ which are atomic or the negation of atomic formulas and satisfy $\mathfrak{M} \models \psi[s]$;

(ii) $\delta^{\mathfrak{M}}_{\lambda,s} = \bigwedge\!\!\!\bigwedge_{\zeta<\lambda} \delta^{\mathfrak{M}}_{\zeta,s}$ if λ is a limit ordinal;

(iii) $\delta^{\mathfrak{M}}_{\alpha+1,s} = \delta^{\mathfrak{M}}_{\alpha,s} \wedge \forall v_n \bigvee\!\!\!\bigvee_{m \in M} \delta^{\mathfrak{M}}_{\alpha,s\frown m} \vee \bigwedge\!\!\!\bigwedge_{m \in M} \exists v_n \delta^{\mathfrak{M}}_{\beta,s\frown m}$.

Let $\delta^{\mathfrak{M}}_{\alpha} = \delta^{\mathfrak{M}}_{\alpha,0}$. The $\delta^{\mathfrak{M}}_{\alpha}$ are the Scott types for the structure $\mathfrak{M}$. They are invariants of $\mathfrak{M}$ (that is, depend only on the isomorphism class of $\mathfrak{M}$), and $\langle \delta^{\mathfrak{M}}_{\beta} : \beta \langle |M|^+ \rangle$ characterises $\mathfrak{M}$ up to $\mathcal{L}_{\infty\omega}(K)$-equivalence. We state the following Lemma, the proof of which is not difficult, and is left to the reader. He may also refer to [13].

Lemma 2.7. (i) *The function $F(\mathfrak{M}, \beta, s) = \delta^{\mathfrak{M}}_{\beta,s}$ is primitive recursive.*

(ii) *Suppose $\varphi \in \mathcal{L}_{\infty\omega}(K)$ is an n-formula of quantifier rank $< \beta$ and $s \in M^n$. Then $\mathfrak{M} \models \varphi[s]$ if and only if*

$$\vdash \forall v_0 \cdots v_{n-1} (\delta^{\mathfrak{M}}_{\beta,s} \to \varphi).$$

(iii) *If $s, t \in M^n$ and $\mathfrak{M} \models \delta^{\mathfrak{M}}_{\beta,s}[t]$, then $\delta^{\mathfrak{M}}_{\beta,t} = \delta^{\mathfrak{M}}_{\beta,s}$.*

A few remarks will be useful before the next theorem. We note that α can be obtained primitive recursively from $\delta^{\mathfrak{M}}_{\alpha}$, uniformly in $\mathfrak{M}$. Also $\{\delta^{\mathfrak{M}}_{\alpha,s\frown m} : m \in M\}$ can be obtained primitive recursively from $\delta^{\mathfrak{M}}_{\alpha+1,s}$. It follows that

$$\{\delta^{\mathfrak{M}}_{\alpha,s} : s \in M^n \quad \text{for some } n\}$$

can be obtained primitive recursively from $\delta^{\mathfrak{M}}_{\gamma}$ and α whenever $\gamma \geq \alpha + \omega$.

Note that by 2.7, $\vdash \delta^{\mathfrak{M}}_{\alpha+1,s} \to \exists v_n \varphi$, where $\mathrm{qr}(\varphi) \leqslant \alpha$ if and only if there is $m \in M$ such that $\vdash \delta^{\mathfrak{M}}_{\alpha,s^\frown m} \to \varphi$. Also, if λ is a limit ordinal and $\mathrm{qr}(\varphi) < \lambda$, $\vdash \delta^{\mathfrak{M}}_{\lambda,s} \to \varphi$ if and only if there is $\xi < \lambda$ such that $\vdash \delta^{\mathfrak{M}}_{\xi,s} \to \varphi$.

Using observations like those above, the reader should verify that there is a PR relation R such that, whenever δ is of the form $\delta^{\mathfrak{M}}_{\alpha,s}$ for some $s \in M^n$ and φ is an n-formula of quantifier rank $< \alpha$, $\vdash \delta \to \varphi$ if and only if $R(\delta, \varphi)$.

Theorem 2.8. *Let A be closed under* PR (χ_K) *functions. If $\mathfrak{M}$ is a K-structure, $A(\mathfrak{M}) \cap V_{U(A)}$ is the primitive recursive closure of*

$$A \cup \{\delta^{\mathfrak{M} \restriction K_0}_{\beta} : \beta \in A,\ K_0 \in A, \text{ and } K_0 \subseteq K\}.$$

Proof. $A(M)$ is PR closed by 2.4 and hence so is $A(\mathfrak{M}) \cap V_{U(A)}$, since $U(TC(F(x))) \subseteq U(TC(x))$ for all x and PR functions F.

By Proposition 2.1, if $K_0 \in A$ and $K_0 \subseteq K$, $\mathfrak{M} \restriction K_0 \in A(\mathfrak{M})$, and so by Lemma 2.7 (i),

$$\delta^{\mathfrak{M} \restriction K_0}_{\beta} \in A(\mathfrak{M}) \cap V_{U(A)}$$

whenever $\beta \in A$. Hence $A(\mathfrak{M}) \cap V_{U(A)}$ contains the PR closure of

$$C = A \cup \{\delta^{\mathfrak{M} \restriction K_0}_{\beta} : \beta \in A,\ K_0 \in A, \text{ and } K_0 \subseteq K\}.$$

Let D be the PR closure of C. Fix $x \in N$. (We fix x because K need not be an element of A — otherwise a uniform construction for all x would be possible.) Let $K_0 = K \cap \mathrm{TC}(x)$ and $\mathfrak{M}_0 = \mathfrak{M} \restriction K_0$. Note that $K_0 \in A$.

We wish to show now that for all $y \in \{x\} \cup \mathrm{TC}(x)$ and $s \in M^{L(x)}$, if $D(\mathfrak{M}, y)(s) \in V_{U(A)}$, $D(\mathfrak{M}, y)(s) \in D$.

By our discussion, earlier in this section, of the meaning of $\models$, we may assume that for all $y \in \{x\} \cup \mathrm{TC}(x)$, if $y \in N$ and y is a function, then $y(w) \in \mathscr{L}_{\infty\omega}(K)$ for all $w \in \mathrm{dom}\ y$. Otherwise we simply replace $y(w)$ by $H_K(y(w))$.

We define first a function γ, which is primitive recursive. $\gamma(y)$ will measure the maximum quantifier rank needed to determine, in a suitable sense, membership in $D(\mathfrak{M}, y)(s)$ when no urelements outside A are involved. Let

$$\lambda = \bigcup \{L(y) : y \in \mathrm{TC}(x)\}.$$

Since A is PR-closed, $\lambda \in A$. $\lambda \leqslant \omega$ and is needed in the development below only if $\omega \notin A$. (If A were pure, the theorem would be trivial in this case; however, we have permitted A to contain urelements. A may therefore contain an infinite set and not ω.) Let

$$\gamma(y) = \begin{cases} 0 & \text{if } y \in \omega,\ y \in U \text{ or } \neg y \in N \\ \sup\{\mathrm{qr}(y(w)) + \gamma(w) + \lambda + 1 : w \in \mathrm{dom}\ y\} & \text{otherwise} \end{cases}$$

Let

$$\gamma_0 = \sup\{\gamma(y)+1 : y \in \mathrm{TC}(x)\}.$$

By remarks above

$$S = \{\langle y, S_y\rangle : y \in \mathrm{TC}(x) \cup \{x\}\} \in D$$

where

$$S_y = \{\delta^{\mathfrak{M}}_{\gamma(\dot{y}),s} : s \in M^{L(y)}\}.$$

For $y \in \{x\} \cup \mathrm{TC}(x)$, we now define a function $\dot{y} : S_y \to V$, as follows:

$$\dot{m}(\delta) = 0 \quad \text{for all } \delta \in S_m \quad \text{if } m \in \omega, \quad \text{or} \quad \neg m \in N$$

$$\dot{y}(\delta) = y \quad \text{for all } \delta \in S_y \quad \text{if } y \in U$$

$$\dot{y}(\delta) = \{\dot{w}(\delta') : w \in \operatorname{dom} y,\ \delta' \in S_w, \text{ and } \vdash \delta(0) \to \exists v^1(y(w) \wedge \delta'(1))\}$$

otherwise, where v^1 is a sequence of the variables v_{1i}, for $i \in L(w)$.

We note that there is a PR function H such that $H(S, y, \delta) = \dot{y}(\delta)$ for $y \in \{x\} \cup \mathrm{TC}(x)$, and $\delta \in S_y$.

We now claim that if $y \in \mathrm{TC}(x) \cup \{x\}$ and $s \in M^{L(y)}$ satisfy $D(\mathfrak{M}, y)(s) \in V_{U(A)}$, then $\dot{y}(\delta^{\mathfrak{M}}_{\gamma(\dot{y}),s}) = D(\mathfrak{M}, y)(s)$. In particular if $D(\mathfrak{M}, x)(s) \in V_{U(A)}$ for $s \in M^{L(x)}$, then $\dot{x}(\delta^{\mathfrak{M}}_{\gamma(\dot{x}),s}) = D(\mathfrak{M}, x)(s)$. So by the claim and the remark immediately above the theorem is proved. We prove the claim by induction on y.

Since $M \cap U(A) = 0$, $D(\mathfrak{M}, y)(s)$ is not in $V_{U(A)}$ if $y \in \omega$, so the claim is true vacuously. If $y \in U$, then $\dot{y}(\delta) = y$ and $D(\mathfrak{M}, y)(0) = y$ so the claim is true. If $\neg y \in N$, it is also true.

Suppose now that y is a nonempty function and that the claim is true for all elements of the domain of y. Suppose also that $s \in M^{L(y)}$ and $D(\mathfrak{M}, y)(s) = a \in V_{U(A)}$. Let $x \in a$. Then there is $w \in \operatorname{dom} y$, $t \in M^{L(w)}$ such that

$$x = D(\mathfrak{M}, w)(t) \quad \text{and} \quad \mathfrak{M} \models y(w)[s, t].$$

By our choice of K_0,

$$a = D(\mathfrak{M}_0, y)(s), \quad x = D(\mathfrak{M}_0, w)(t), \quad \text{and} \quad \mathfrak{M}_0 \models y(w)[s, t].$$

By the induction hypothesis,

$$x = \dot{w}(\delta^{\mathfrak{M}_0}_{\gamma(\dot{w}),t})$$

since

$$x \in V_{U(A)} \quad \text{and} \quad \delta^{\mathfrak{M}_0}_{\gamma(\dot{w}),t} \in S_w.$$

By the definition of $\gamma(y)$, the formula

$$\exists v^1(y(w) \wedge \delta^{\mathfrak{M}_0}_{\gamma(\dot{w}),t}(1))$$

has quantifier rank $< \gamma(y)$. But

$$\mathfrak{M}_0 \models (\gamma(w) \wedge \delta^{\mathfrak{M}}_{\gamma(\dot{w}),t}(1))[s, t]$$

and hence

$$\mathfrak{M}_0 \models \exists v^1(\gamma(w) \wedge \delta^{\mathfrak{M}}_{\gamma(\dot{w}),t}(1))[s].$$

By Lemma 2.7 (ii),

$$\vdash \delta^{\mathfrak{M}}_{\gamma(\dot{y}),s} \to \exists v^1(\gamma(w) \wedge \delta^{\mathfrak{M}}_{\gamma(\dot{w}),t}(1)).$$

But this means that $x \in \dot{y}(\delta^{\mathfrak{M}}_{\gamma(\dot{y}),s})$. So $a \subseteq \dot{y}(\delta^{\mathfrak{M}}_{\gamma(\dot{y}),s})$.

We now establish the converse inclusion. Let $x \in \dot{y}(\delta^{\mathfrak{M}}_{\gamma(\dot{y}),s})$. Then there is $w \in \operatorname{dom} y$ and $\delta' \in S_w$, such that

$$\vdash \delta^{\mathfrak{M}}_{\gamma(\dot{y}),s} \to \exists v^1(\gamma(w) \wedge \delta'(1)), \quad \text{and} \quad x = \dot{w}(\delta').$$

But

$$\mathfrak{M}_0 \models \delta^{\mathfrak{M}}_{\gamma(\dot{y}),s}[s]$$

so

$$\mathfrak{M}_0 \models (\gamma(w) \wedge \delta'(1))[s, t] \quad \text{for some } t \in M^{L(w)}.$$

By Lemma 2.7 (iii), $\delta' = \delta^{\mathfrak{M}}_{\gamma(\dot{w}),t}$ and by the induction hypothesis, $\dot{w}(\delta') = D(\mathfrak{M}, w)(t)$. But clearly $\mathfrak{M} \models y(w)[s, t]$, so

$$x = D(\mathfrak{M}, w)(t) \in D(\mathfrak{M}, y)(s).$$

Hence the converse inclusion is proved, and $a \subseteq \dot{y}(\delta^{\mathfrak{M}}_{\gamma(\dot{y}),s})$.

Hence the claim and the theorem are proved. □

Corollary 2.9. *Let A be a pure set closed under* PR (χ_K) *functions and let $\mathfrak{M}$ be a K-structure with universe M a set of urelements. Then the pure part of $A(\mathfrak{M})$ is the primitive recursive closure of A and the set of $\delta^{\mathfrak{M} \upharpoonright K_0}_{\beta}$ for $\beta \in A$ and $K_0 \in A$ satisfying $K_0 \subseteq K$.*

Proof. This is simply 2.8 in the case $U(A) = 0$. □

In two crucial places in this section we have made use of the assumption that M is a set of urelements disjoint from A. It was needed in Proposition 2.2, where it was used to assure that the membership relation in M and between elements of M and of A could be "computed" in A. In this case the relation is trivial. But this "computation" would be equally possible if M were an element of A. In this case, the full analysis works for the structure $\mathfrak{M}' = (\mathfrak{M}, m)_{m \in M}$. The constants $\dot{m}$ are needed to reduce formulas of the form $D(\mathfrak{M}, 0)(s) = a$ where $a \in A$.

In [16], Ressayre performed essentially the analysis of 2.1 through 2.3, and 2.6 for the case of $K, M \in A$ and $\mathfrak{M}'$ as above. (The constants are assumed to be the $\dot{m}$'s of Section 1.) He worked however with the assumption that A is admissible (and used this assumption).

Theorem 2.8 is a generalisation of a theorem of Nadel–Stavi [14], which yields the same conclusion when A is L_α and α is the least ordinal for which $L_\alpha(\mathfrak{M})$ is admissible. In this case M is assumed to be a set of urelements. $L_\alpha(\mathfrak{M})$ in this case is called HYP ($\mathfrak{M}$), a notion due to Barwise. (See, in particular, [3].) K is assumed by Barwise to be finite, since he is interested mainly in the case $\alpha = \omega$. Nadel–Stavi [14] loosens this requirement somewhat. Above it has been loosened about as far as possible. In fact, a careful examination of the proof shows that the sole restriction on K is the hypothesis that if $a \in A$, then there is $b \in A$ such that $a \cap K \subseteq b \subseteq K$. Theorem 2.8 in its full generality does not appear to follow from the theorem of Nadel–Stavi.

3. +-admissible and special structures

We present in this section a notion generalising admissibility. We shall consider structures $\mathscr{A} = (\mathbf{A}, R_1, \ldots, R_n)$ where each R_i is a unary relation on A. The language for the structure shall be $L(\in, \mathrm{U}, P_1, \ldots, P_n)$ where, for $1 \leqslant i \leqslant n$, R_i interprets P_i.

While the whole theory could easily be reduced to the case $n = 1$, it will sometimes be convenient to have the notions as stated below for $n \geqslant 1$. Many proofs will be given only in the simpler case $n = 1$, however. The more general proofs are more difficult only notationally.

Definition 3.1. $\mathscr{A} = (\mathbf{A}, R_1, \ldots, R_n)$ is +-admissible if it satisfies the following conditions:

(i) if $x, y \in A$, then $\{x, y\} \in A$; $\phi \in A$;
(ii) if $B \in A$ and B is a set of sets, $\bigcup B \in A$;
(iii) if X is Δ_0 on $\mathbf{A}$ and $a \in A$, then $X \cap a \in A$;
(iv) if X is Δ_0 on $\mathbf{A}$ and $a \in A$, and

$$\forall x \in a\, \exists y_0, y_1, \ldots, y_n \in A[y_1 \subseteq R_1 \wedge \cdots \wedge y_n \subseteq R_n \wedge Xxy_0y_1 \cdots y_n],$$

then there are $a_0, a_1, \ldots, a_n \in A$ such that $a_i \subseteq R_i$ for $1 \leqslant i \leqslant n$ and

$$\forall x \in a\, \exists y_0, y_1, \ldots, y_n \in a_0[y_1 \subseteq a_1 \wedge \cdots \wedge y_n \subseteq a_n \wedge Xxy_0y_1 \cdots y_n].$$

It follows easily from (iv) that if $\mathscr{A}$ is +-admissible, then $\mathbf{A}$ satisfies the Δ_0-collection axiom, and therefore by (i), (ii), (iii), and (iv), $\mathbf{A}$ is admissible.

The notion of +-admissible structure is more general than the notion of admissible set or structure. The motivating idea for this notion is consideration of structures $\mathscr{A} = (\mathbf{A}, R)$ where R is Σ on $\mathbf{A}$. We shall prove below that all such structures are +-admissible. The converse is not true. (A counterexample may be found, for example, by a forcing argument.)

Proposition 3.1. *Let $\mathscr{A} = (\mathbf{A}, R_1, \ldots, R_n)$. Suppose $\mathscr{A}$ satisfies (i), (ii), (iii) of Definition 3.1, and (iv) for Δ_0 relations X on $\mathbf{A}$ with the additional property that*

$$y_0 \subseteq z_0 \wedge \cdots \wedge y_n \subseteq z_n \wedge Xxy_0 \cdots y_n \rightarrow Xxz_0 \cdots z_n.$$

Then $\mathscr{A}$ is $+$-admissible.

Proof. Suppose $\mathscr{A}$ is as in the hypotheses, let X be Δ_0 on $\mathbf{A}$, $a \in A$, and suppose

$$\forall x \in a \exists y_0, y_1, \ldots, y_n \in A[y_1 \subseteq R_1 \wedge \cdots \wedge y_n \subseteq R_n \wedge Xxy_0 \cdots y_n].$$

Let $X'xy_0 \cdots y_n$ be the relation

$$(\exists z_0, z_1, \ldots, z_n \in y_0)(z_1 \subseteq y_1 \wedge \cdots \wedge z_n \subseteq y_n \wedge Xxz_0z_1 \cdots z_n).$$

Then X' is Δ_0 on $\mathbf{A}$ and

$$\forall x \in a \exists y_0, y_1, \ldots, y_n \in A[y_1 \subseteq R_1 \wedge \cdots \wedge y_n \subseteq R_n \wedge X'xy_0 \cdots y_n].$$

Hence there are $a_0', a_1', \ldots, a_n' \in A$ such that $a_i' \subseteq R_i$ for $1 \leqslant i \leqslant n$ and

$$\forall x \in a \exists y_0, y_1, \ldots, y_n \in a_0'[y_1 \subseteq a_1' \wedge \cdots \wedge y_n \subseteq a_n' \wedge X'xy_0 \cdots y_n].$$

Let $a_0 = \mathrm{TC}(a_0')$. Then $a_0 \in A$ and it is easy to check that

$$\forall x \in a \exists y_0, y_1, \ldots, y_n \in a_0[y_1 \subseteq a_1' \wedge \cdots \wedge y_n \subseteq a_n' \wedge Xxy_0 \cdots y_n].$$

Hence $\mathscr{A}$ is $+$-admissible as required. $\square$

A relation $Xxy_0 \cdots y_n$ satisfying the property in the hypothesis shall be called "increasing in $y_0, \ldots, y_n$".

We shall call (iv) of Definition 3.1 the $+$-collection axiom schema. Proposition 3.1 shows that to establish the $+$-admissibility of a structure, we need only verify this axiom schema for X increasing in $y_0, \ldots, y_n$.

If X is a relation on A we say that X is Σ^+ on $\mathscr{A} = (\mathbf{A}, R_1, \ldots, R_n)$ if there is a relation X' Δ_0 on $\mathbf{A}$ such that, for all $x_1, \ldots, x_k \in A$

$$Xx_1 \cdots x_k \leftrightarrow \exists y_0, y_1, \ldots, y_n \in A[y_1 \subseteq R_1 \wedge \cdots \wedge y_n \subseteq R_n \wedge X'x_1 \cdots x_ky_0y_1 \cdots y_n].$$

Thus Σ^+ relations on $+$-admissible structures are generalisations of Σ_1 relations on admissible structures. Proposition 3.3 below shows that, among other things, they satisfy quite similar closure properties.

Proposition 3.2. *Let $\mathscr{A} = (\mathbf{A}, R_1, \ldots, R_n)$ be $+$-admissible and let X be Σ^+ on $\mathscr{A}$. Then there is $X'x_1 \cdots x_ky_0 \cdots y_n$ which is Δ_0 on $\mathbf{A}$, increasing in $y_0, \ldots, y_n$, and which satisfies:*

(1) *$Xx_1 \cdots x_k$ if and only if*

$$\exists y_0, y_1, \ldots, y_n \in A[y_1 \subseteq R_1 \wedge \cdots \wedge y_n \subseteq R_n \wedge X'x_1 \cdots x_ky_0y_1 \cdots y_n].$$

Proof. Since X is Σ^+ on $\mathscr{A}$ there is X_0 Δ_0 on $\mathbf{A}$ such that

$$Xx_1 \cdots x_k \leftrightarrow \exists y_0 y_1 \cdots y_n \in A[y_1 \subseteq R_1 \wedge \cdots \wedge y_n \subseteq R_n \wedge X_0 x_1 \cdots x_k y_0 \cdots y_n].$$

Let $X' x_1 \cdots x_k y_0 \cdots y_n$ be the relation

$$(\exists z_0 \cdots z_n \in y_0)(z_1 \subseteq y_1 \wedge \cdots \wedge z_n \subseteq y_n \wedge X_0 x_1 \cdots x_k z_0 \cdots z_n).$$

Then it is easily verified that X' satisfies the conclusion of the theorem. □

Proposition 3.3. *Let* $\mathscr{A} = (\mathbf{A}, R_1, \ldots, R_n)$ *be* $+$*-admissible.*

(i) R_i *is* Σ^+ *on* $\mathscr{A}$ *for* $1 \leq i \leq n$.

(ii) *If* X *is* Δ_0 *on* $\mathbf{A}$, *then* X *is* Σ^+ *on* $\mathscr{A}$.

(iii) *If* X *and* Y *are* Σ^+ n*-ary relations on* $\mathscr{A}$, *so are* $X \cap Y$ *and* $X \cup Y$.

(iv) *If* $Xx_0 \cdots x_n$ *is* Σ^+ *on* $\mathscr{A}$, *so are* Y, Z *where*

$$Yx_1 \cdots x_n \leftrightarrow \exists x_0 \in A\, Xx_0 x_1 \cdots x_n$$

and

$$Zzx_1 \cdots x_n \leftrightarrow \forall x \in z Xxx_1 \cdots x_n.$$

Proof. (i) $R_i x \leftrightarrow \exists y_1 \in A[y_1 \subseteq R_i \wedge x \in y_1]$. (ii), (iii) are trivial. In (iv) it is trivial to show that Y is Σ^+ on $\mathscr{A}$. We consider Z. For convenience we shall assume that $n = 1$ and $R_1 = R$. Since X is Σ^+ on $\mathscr{A}$ there is X' Δ_0 on $\mathbf{A}$ such that

$$Xx_0 x_1 \cdots x_n \leftrightarrow \exists y_0, y_1 \in A[y_1 \subseteq R \wedge X' x_0 x_1 \cdots x_n y_0 y_1].$$

We may further assume by Proposition 3.2 that X' is increasing in y_0 and y_1.

We claim that

$$Zzx_1 \cdots x_n \leftrightarrow \exists y_0, y_1 \in A[y_1 \subseteq R \wedge \forall x_0 \in z \exists y_0' \in y_0 X' x_0 x_1 \cdots x_n y_0' y_1].$$

This will complete the proof. $\leftarrow$ is trivial to verify. Suppose that $Zzx_1 \cdots x_n$. Then by definition of Z,

$$\forall x_0 \in z \exists y_0, y_1 \in A[y_1 \subseteq R \wedge X' x_0 x_1 \cdots x_n y_0 y_1].$$

By (iv) of Definition 3.1, there are a_0, a_1 A such that $a_1 \subseteq R$ and

$$\forall x_0 \in z \exists y_0, y_1 \in a_0[y_1 \subseteq a_1 \wedge X' x_0 x_1 \cdots x_n y_0 y_1].$$

Hence, as X' is increasing in y_0, y_1,

$$\forall x_0 \in z \exists y_0 \in a_0 X' x_0 x_1 \cdots x_n y_0 a_1.$$

But this clearly proves the claim. □

We may define a Σ^+ formula of $L(\varepsilon, \mathrm{U}, P_1, \ldots, P_n)$ as one built up from Δ_0 formulas of $L(\varepsilon, \mathrm{U})$ (which contain no occurrence of a P_i) and formulas of the form $P_j v_i$ by use of $\wedge$, $\vee$, $\exists v_i$, and $\forall v_i \varepsilon v_j$, $i, j \in \omega$. It is clear that every Σ^+ relation is defined by a Σ^+ formula (with parameters) and Proposition 3.2 shows that the converse is also true. It also shows that for each Σ^+ formula φ with free variables among $v_0, \ldots, v_{k-1}$ there is a Δ_0 formula θ of $L(\varepsilon, U, P_1, \ldots, P_n)$, with free variables $v_0, \ldots, v_k$, in which all occurrences of the P_i's are positive, such that for all $+$-admissible $\mathscr{A}$ and all $s \in A^k$,

$$\mathscr{A} \models \varphi[s] \quad \text{if and only if} \quad \mathscr{A} \models \exists v_k\, \theta[s].$$

We do not emphasize this normal form, as $+$-admissible structures need not satisfy the separation axiom for such θ.

A relation X on $\mathscr{A}$ is Δ^+ on $\mathscr{A}$ if both X and its complement in A are Σ^+ on $\mathscr{A}$. The following theorem generalises a familiar property of relations Δ on admissible structures.

Proposition 3.4. *Let $\mathscr{A} = (\mathbf{A}, R_1, \ldots, R_n)$ be $+$-admissible. Let $X \subseteq A$ and suppose X is Δ^+ on $\mathscr{A}$. If $a \in A$, $X \cap a \in A$.*

Proof. We assume again for convenience that $n = 1$ and $R_1 = R$. X and $A - X$ are Σ^+ on $\mathscr{A}$, so there are Y_1, Y_2 Δ_0 on $\mathbf{A}$ such that, for $x \in A$,

$$Xx \leftrightarrow \exists y_0, y_1 \in A[y_1 \subseteq R \wedge Y_1 x y_0 y_1],$$

and

$$\neg Xx \leftrightarrow \exists y_0, y_1 \in A[y_1 \subseteq R \wedge Y_2 x y_0 y_1].$$

Now since $X \cup (A - X) = A$,

$$\forall x \in a\, \exists y_0, y_1 \in A[y_1 \subseteq R \wedge (Y_1 x y_0 y_1 \vee Y_2 x y_0 y_1)].$$

Since $\mathscr{A}$ is $+$-admissible, there are $a_0, a_1 \in A$ such that $a_1 \subseteq R$ and

$$\forall x \in a\, \exists y_0, y_1 \in a_0[y_1 \subseteq a_1 \wedge (Y_1 x y_0 y_1 \vee Y_2 x y_0 y_1)].$$

Let

$$c = \{x \in a : \exists y_0, y_1 \in a_0(y_1 \subseteq a_1 \wedge Y_1 x y_0 y_1)\}.$$

Clearly $c \subseteq X \cap a$. If $x \in a - c$, there are $y_0, y_1 \in a_0$ such that $y_1 \subseteq a_1$ and $Y_2 x y_0 y_1$, so $x \in (A - X)$. Hence $c = A \cap a$. But $c \in A$ by (iii) of Definition 3.1. □

It is possible also to prove certain natural collection schema for $+$-admissible sets. The first part of the Proposition below is the Σ^+-collection schema. The second part is a useful strengthening. For admissible sets and relations, (ii) below follows easily from (i). For $+$-admissible structures, however, a small trick is involved.

Proposition 3.5 *Let $\mathscr{A} = (\mathbf{A}, R_1, \ldots, R_n)$ be $+$-admissible, let X be Σ^+ on $\mathscr{A}$ and let $a \in A$.*

(i) *If $\forall x \in a\, \exists y \in A\, Xxy$, then there is $b \in A$ such that $\forall x \in a\, \exists y \in b\, Xxy$.*

(ii) *If Y is Σ^+ on $\mathscr{A}$ and $\forall x \in a\, \exists y \in Y(Xxy)$ there is $b \in A$ such that $b \subseteq Y$ and $\forall x \in a\, \exists y \in b\, Xxy$.*

Proof. We prove only (ii), as (i) is a special case. Again for simplicity, suppose that $n=1$ and $R=R_1$.

Since X, Y are Σ^+ on $\mathscr{A}$, there are X_0, Y_0 Δ_0 on $\mathbf{A}$ such that, for $x, y \in A$,

(1) $\quad Yy \leftrightarrow \exists y_0, y_1 \in A[y_1 \subseteq R \wedge Y_0 yy_0 y_1]$

(2) $\quad Xxy \leftrightarrow \exists z_0, z_1 \in A[z_1 \subseteq R \wedge X_0 xyz_0 z_1]$.

By Proposition 3.2, we may also assume that, if $y_0 \subseteq y_1$ and $z_0 \subseteq z_1$, then

(3) $\quad Y_0 yy_0 z_0 \rightarrow Y_0 yy_1 z_1 \quad$ for all $y \in A$

and

(4) $\quad X_0 xyy_0 z_0 \rightarrow X_0 xyy_1 z_1 \quad$ for all $x, y \in A$

Suppose now that $a \in A$ and

$$\forall x \in a\, \exists y \in Y(Xxy).$$

Then, in A,

$$\forall x \in a \exists y, y_0, y_1, z_0, z_1[y_1 \subseteq R \wedge z_1 \subseteq R \wedge Y_0 yy_0 y_1 \wedge X_0 xyz_0 z_1].$$

By (3) and (4), in A still,

$$\forall x \in a\, \exists y_0, y_1[y_1 \subseteq R \wedge (\exists y \in y_0)(Y_0 yy_0 y_1 \wedge X_0 xyy_0 y_1)].$$

By $+$-admissibility, there are a_0', $a_1 \in A$ such that $a_1 \subseteq R$ and

$$\forall x \in a\, \exists y_0, y_1 \in a_0'[y_1 \subseteq a_1 \wedge (\exists y \in y_0)(Y_0 yy_0 y_1 \wedge X_0 xyy_0 y_1)].$$

Let $a_0 = \mathrm{TC}(a_0')$. Then $a_0 \in A$, and

(5) $\quad \forall x \in a\, \exists y, y_0 \in a_0[Y_0 yy_0 a_1 \wedge X_0 xyy_0 a_1]$.

Let

$$c = \{y \in a_0 . \exists y_0 \in a_0\, Y_0 yy_0 a_1\}.$$

Then $c \in A$ by Δ_0-separation. Further, by (1), $c \subseteq Y$. And finally, by (5) and (2),

$$\forall x \in a\, \exists y \in c\, Xxy$$

as required. $\square$

The reader should now find it easy to verify that a structure $(\mathbf{A}, R)$ is fully admissible if and and only if the structure $(\mathbf{A}, R, A-R)$ is $+$-admissible.

Proposition 3.6. *Let $\mathscr{A} = (\mathbf{A}, R_1, \ldots, R_m)$ be $+$-admissible. Suppose $G : A^{n+2} \to A$ for some $n \in \omega$ and that G is Σ^+ on $\mathscr{A}$. Then there is $F : A^{n+2} \to A$ which is Σ^+ on $\mathscr{A}$ and satisfies*

(*) $\quad F(x, z_1, \ldots, z_n) = G(\bigcup\{F(u, z_1, \ldots, z_n) : u \in x\}, x, z_1, \ldots, z_n)$
for all $x, z_1, \ldots, z_n \in A$.

Proof. We shall give only a sketch as the proof is essentially that used in set theory.

Define a partial function F' on A^{n+1} by the definition:

$$(\dagger)\qquad F'(x, z_1, \ldots, z_n) = y \leftrightarrow (\exists f \in A)[\mathrm{Trans}\,(\mathrm{dom}\, f) \wedge x \in \mathrm{dom}\, f \wedge f(x) = y$$
$$\wedge (\forall x' \in \mathrm{dom}\, f)(f(x') = G(\bigcup \{f(u) : u \in x'\}, x', z_1, \ldots, z_n))$$

The usual argument shows that F' is single valued and therefore in fact a function as claimed.

Furthermore, were F' defined on A^{n+1}, F' would be the F of the theorem, as the definition $(\dagger)$ above is Σ^+ on $\mathscr{A}$, and F' satisfies the recursion equation $(*)$ for $x, z_1, \ldots, z_n A$.

So it is enough to show that F' is total. But this is easily proved by induction on the rank of x. We leave this to the reader. □

Corollary 3.7. *Let $\mathscr{A} = (\mathbf{A}, R_1, \ldots, R_n)$ be $+$-admissible. Suppose $H_1, \ldots, H_m$ are set functions such that $H_i \restriction A$ is Δ^+ on $\mathscr{A}$ (and hence maps A into A). Then if F is* PR $(H_1, \ldots, H_m)$, *A is closed under $F \restriction A$ and $F \restriction A$ is Δ^+ on $\mathscr{A}$.*

Proof. This can be proven easily by induction on PR $(H_1, \ldots, H_m)$. The only nontrivial step is primitive recursion, but it follows directly from Proposition 3.6. □

Proposition 3.8. *Let $\mathscr{A} = (\mathbf{A}, R_1, \ldots, R_n)$ be $+$-admissible and let X be Σ^+ on $\mathscr{A}$. Then $\mathscr{A}_X = (\mathbf{A}, R_1, \ldots, R_n, X)$ is $+$-admissible.*

Proof. We shall give the proof in the simple case that $n = 1$, $R = R_1$, $X \subseteq A$ as this contains all the non-notational difficulty of the general case.

Suppose X_0 is Δ_0 on $\mathbf{A}$, $a \in A$, and

$$(*)\qquad \forall x \in a \exists y_0, y_1, y_2 \in A[y_1 \subseteq X \wedge y_2 \subseteq R \wedge X_0 x y_0 y_1 y_2].$$

Now $Y = \{a \in A : a \subseteq X\}$ is Σ^+ on $\mathscr{A}$. From $(*)$,

$$\forall x \in a\, \exists y_1 \in Y[\exists y_0, y_2 \in A(y_2 \subseteq R \wedge X_0 x y_0 y_1 y_2)].$$

By Proposition 3.5, there is $b \in A$, $b \subseteq Y$ such that

$$\forall x \in a\, \exists y_1 \in b[\exists y_0, y_2 \in A(y_2 \subseteq R \wedge X_0 x y_0 y_1 y_2)].$$

Let $a_1 = \bigcup b$. Then $a_1 \subseteq X$ and

$$\forall x \in a\, \exists y_0 y_1 y_2 [y_2 \subseteq R \wedge X_0 x y_0 y_1 y_2 \wedge y_1 \subseteq a_1].$$

a_0 and a_2 may now be found simply, by the $+$-admissibility of $\mathscr{A}$, thereby completing the argument. □

It is a simple consequence of Proposition 3.8 that if $\mathbf{A}$ is admissible and R is Σ on $\mathbf{A}$, then $\mathscr{A}=(\mathbf{A}, R)$ is $+$-admissible.

Admissibility for structures $(\mathbf{A}, R_1, \ldots, R_n)$ can be characterized in terms of a reflection schema:

$$\theta \to \exists u\,(\text{“Trans}\,(u) \wedge u_1, \ldots, u_n \in u\text{”} \wedge \theta^{(u)})$$

where θ is a Σ formula of $L(\varepsilon, \mathrm{U}, P_1, \ldots, P_n)$, u is a variable not appearing in θ, $u_1, \ldots, u_n$ are all the free variables of θ, and $\theta^{(u)}$ is the result of replacing in θ all unbounded quantifications $\forall v_i, \exists v_i$ by $\forall v_i \varepsilon u$, $\exists v_i \varepsilon u$, respectively, for $i \in \omega$. A similar possibility is available for $+$-admissibility.

Definition 3.2. Let θ be a formula of $L(\varepsilon, \mathrm{U}, P_1, \ldots, P_n)$ and $u_1, \ldots, u_n$ be variables. Then $\theta(P_1, \ldots, P_n/u_1, \ldots, u_n)$ (which we shall also write $\theta(P/u)$) is the result of replacing each occurrence in θ of $P_i v_j$ by $v_j \varepsilon u_i$ (and changing bound variables to avoid clashing).

Proposition 3.9. *Let $\mathscr{A}=(\mathbf{A}, R_1, \ldots, R_n)$ and suppose A is admissible. Then the following are equivalent:*

(A) *$\mathscr{A}$ is $+$-admissible*

(B) *For every Σ^+ formula θ of $L(\varepsilon, \mathrm{U}, P_1, \ldots, P_n)$,*

$$\mathscr{A} \models \theta \to \exists u_1, \ldots, u_n (u_1 \subseteq P_1 \wedge \cdots \wedge u_n \subseteq P_n \wedge \theta(P/u))$$

where $u_1, \ldots, u_n$ are variables not appearing in θ.

Proof. We sketch the proof for $n=1$ and $P=P_1$. $+$-collection is a simple consequence of (B), so (B)$\to$(A). For the other direction, assume (A). As $\mathbf{A}$ is admissible it is a model of KPU.

We shall leave it to the reader to prove by induction on Σ^+ formulas θ that there is a Δ_0 formula $\theta_1(u_0, u)$ (of $L(\varepsilon, \mathrm{U})$, possibly having free variables besides u_0 and u) such that

(1) $\quad \theta(P/u) \leftrightarrow \exists u_0 \theta_1(u_0, u)$

is a theorem of KPU, and

(2) $\quad \mathscr{A} \models \theta \to \exists u_0, u\,(u \subseteq P \wedge \theta_1(u_0, u))$.

This will complete the proof, as $\mathbf{A} \models \mathrm{KPU}$, so (1) and (2) may be combined to show that

$$\mathscr{A} \models \theta \to \exists u\,(u \subseteq P \wedge \theta(P/u)). \quad \square$$

We are now in a position to prove the “Barwise Compactness Theorem” for $+$-admissible structures.

Theorem 3.10. *Let $\mathscr{A}=(\mathbf{A}, R_1, \ldots, R_n)$ be $+$-admissible and countable. Suppose T is a Σ^+ on $\mathscr{A}$ subset of $\mathscr{L}_{\infty\omega}$ and that every $\mathbf{A}$-finite subset of T has a model. Then T has a model.*

Proof. Suppose that $\mathbb{A}^{\mathscr{C}} \in A$ and

(*) $\quad \forall \varphi \in \Phi\, \exists s \in A[s \subseteq T \wedge s \vdash \varphi]$

By 1.2 and 3.7, it is sufficient to show that

(†) $\quad \exists s_0 \in A(s_0 \subseteq A \wedge s_0 \vdash \mathbb{A}\Phi)$.

Now T is Σ^+ on $\mathscr{A}$, hence so is $\{s \in A : s \subseteq T\}$. Also $\{(\varphi, s) : s \in A \wedge s \vdash \varphi\}$ is Σ^+ on $\mathscr{A}$ by 1.1. Hence, by Proposition 3.5, we may infer from (*) that there is $b \in A$ such that $b \subseteq T$ and $\forall \varphi \in \Phi\, \exists s \in b(s \vdash \varphi)$. But then if $s_0 = \bigcup b$, $s_0 \in A$ and satisfies (†) above as required. □

Definition 3.3. Let $\mathscr{A} = (\mathbf{A}, R_1, \ldots, R_n)$ and suppose A is PR-closed. Let C be a set and $S_1, \ldots, S_k$ unary relations on C.

(i) C is $\mathscr{A}$-bounded if for every $c \in C$ and $X \subseteq A$ such that X is Σ^+ on $\mathscr{A}$ and $c \subseteq X$, there is $a \in A$ such that $c \subseteq a \subseteq X$.

(ii) $\mathscr{C} = (\mathbf{C}, S_1, \ldots, S_k)$ is $\mathscr{A}$-special if $C \supseteq A$, C is $\mathscr{A}$-bounded and $(\mathscr{C}, \mathscr{A})$ is $+$-admissible.

If $\mathscr{C}$ is $\mathscr{A}$-special, $\mathscr{C}$, as a reduct of a $+$-admissible structure, must be $+$-admissible itself. More remarkably, it follows also that $\mathscr{A}$ is $+$-admissible. For suppose that $\mathscr{C}$ is $\mathscr{A}$-special where $\mathscr{A} = (\mathbf{A}, R)$, X is Δ_0 on $\mathbf{A}$, $a \in A$, and

$$\forall x \in a\, \exists y_0, y_1 \in A[y_1 \subseteq R \wedge Xxy_0y_1].$$

It follows easily from the $+$-admissibility of $(\mathscr{C}, \mathscr{A})$ that there are $c_0, c_1 \in C$ such that $c_0 \subseteq A$, $c_1 \subseteq R$, and

$$\forall x \in a\, \exists y_0, y_1 \in c_0[y_1 \subseteq c_1 \wedge Xxy_0y_1].$$

But as C is $\mathscr{A}$-bounded, there are $a_0, a_1 \in A$ such that $c_0 \subseteq a_0$, $c_1 \subseteq a_1 \subseteq R$, and

$$\forall x \in a\, \exists y_0, y_1 \in a_0[y_1 \subseteq a_1 \wedge Xxy_0y_1].$$

But this verifies $+$-collection in $\mathscr{A}$. The other parts of the definition follow from the fact that A is PR closed.

Proposition 3.11 (i) *$(\mathscr{C}, \mathscr{A})$ is $+$-admissible if and only if $(\mathscr{C}, A, R_1, \ldots, R_n)$ is $+$-admissible.*

(ii) *If $\mathbf{A}$ is admissible with ordinal α and $x \in A$, then $\mathbf{A}$ is $\mathbf{L}_\alpha(x)$-special.*

(iii) *If $\mathbf{A}$ is admissible and $X \in A$ is a set of urelements, then $\mathbf{A}$ is $\mathbf{A} \cap \mathbf{V}_X$-special.*

Proof (i) This follows by Proposition 3.8 from the fact that $\in \restriction A^2$ is Σ^+ on $(\mathscr{C}, A, R_1, \ldots, R_n)$.

(ii) Since $L_\alpha(x)$ is Σ on A, $(\mathbf{A}, L_\alpha(x))$ is $+$-admissible. Let us verify that A is $L_\alpha(x)$-bounded. Let $c \in A$ satisfy $c \subseteq L_\alpha(x)$. Then

$$\forall y \in c\, \exists \beta \in A(y \in L_\beta(x))$$

By Σ-collection in $\mathbf{A}$ there is $\beta_0 \in A$ such that

$$\forall y \in c\, \exists \beta < \beta_0(y \in L_\beta(x))$$

But then $c \subseteq L_{\beta_0}(x) \in L_\alpha(x)$, as required.

(iii) This is left to the reader. □

Proposition 3.12. (i) *Let A be PR-closed and suppose B is $\mathbf{A}$-bounded and $B \supseteq A$. Then B and A have the same ordinals.*

(ii) *Suppose $\mathscr{B}$ is $\mathscr{A}$-special and $\mathscr{C}$ is $(\mathscr{B}, \mathscr{A})$-special. Then $\mathscr{C}$ is $\mathscr{A}$-special.*

Proof. (i) This follows easily from the fact that On is Δ_0.

(ii) If $\mathscr{C}$ is $(\mathscr{B}, \mathscr{A})$-special, $(\mathscr{C}, (\mathscr{B}, \mathscr{A}))$ is $+$-admissible, and hence so is $(\mathscr{C}, \mathscr{A})$. If $X \subseteq A$ is Σ^+ on $\mathscr{A}$ and $c \in C$ is such that $c \subseteq X$, then X is Σ^+ on $(\mathscr{B}, \mathscr{A})$, so there is $b \in B$ such that $c \subseteq b \subseteq X$. But B is $\mathscr{A}$-bounded, so there is $a \in A$ such that $b \subseteq a \subseteq X$. Hence $c \subseteq a \subseteq X$ as required. □

An admissible set B is a fattening of an admissible set A if $B \supseteq A$ and they have the same ordinals (cf. Nadel [18]). Thus any $\mathbf{A}$-special structure is a fattening of A. Furthermore, by 3.11 (ii), any fattening of an admissible set $L_\alpha(x)$ is $\mathbf{L}_\alpha(\mathbf{x})$-special. We shall show in Section 5, however, that there are A, B admissible such that B is a fattening of A, but is not $\mathbf{A}$-special.

Proposition 3.13. *Suppose $\mathscr{C} = (\mathbf{C}, S)$ is $\mathscr{A}$-special, where $\mathscr{A} = (\mathbf{A}, R)$. Let h be a partial function on A, Σ^+ on $(\mathscr{C}, \mathscr{A})$ and such that* dom h *is Σ^+ on $\mathscr{A}$. Let B be*

$$\bigcup \{L_\alpha(\{a, h \restriction b\}) : a, b \in A, b \subseteq \operatorname{dom} h\}.$$

Let $\mathscr{B} = (\mathbf{B}, h)$.

(i) *$\mathscr{C}$ is $\mathscr{B}$-special*

(ii) *$\mathscr{B}$ is $\mathscr{A}$-special.*

Proof. We prove only (ii). B clearly is transitive, closed undertaking pairs, union of families of sets, and $\mathbf{B}$ satisfies Δ_0-separation. B is also $\mathscr{A}$-bounded, as C is. It is enough to show that $(\mathbf{B}, h, \mathbf{A}, R)$ is $+$-admissible.

Let X be Δ_0 on $\mathbf{B}$, $a \in \mathbf{B}$, and suppose that

$$\forall x \in a\, \exists y_0, y_1, y_2, y_3 \in \mathbf{B}[y_1 \subseteq h \wedge y_2 \subseteq A \wedge y_3 \subseteq R \wedge Xxy_0y_1y_2y_3].$$

By 3.5 (actually a slight variant) applied to $(\mathscr{C}, \mathscr{A})$ (blurring the distinction between X and any of its Δ_0 extensions to $\mathbf{C}$), we get $c_0 \in C$ and then by the

$+$-admissibility of $(\mathscr{C}, \mathscr{A})$ we get $c_1, c_2, c_3 \in C$ such that $c_0 \subseteq B$, $c_1 \subseteq h$, $c_2 \subseteq A$, $c_3 \subseteq R$ and

$$\forall x \in a\, \exists y_0, y_1, y_2, y_3 \in c_0[y_1 \subseteq c_1 \wedge y_2 \subseteq c_2 \wedge y_3 \subseteq Xxy_0y_1y_2y_3].$$

By the $\mathscr{A}$-boundedness of C we may assume $c_2, c_3 \in A$. It is enough to show there are $d_0, d_1 \in B$ such that $c_0 \subseteq d_0$ and $c \subseteq d_1 \subseteq h$. We shall prove the existence of d_1. d_0 can be found similarly.

Since $c_1 \subseteq h$,

$$\forall x \in c_1\, \exists a \in \operatorname{dom} h[x = \langle a, h(a)\rangle].$$

Hence, there is $c' \in C$ such that $c' \subseteq \operatorname{dom} h$ and

$$\forall x \in c_1\, \exists a \in c'[x = \langle a, h(a)\rangle].$$

C is $\mathscr{A}$-bounded so there is $d' \in A$ such that $c' \subseteq d' \subseteq \operatorname{dom} h$. Let $d_1 = h \restriction d'$. Then $d_1 \in B$ and $c_1 \subseteq d_1$, as required. Hence $\mathscr{B} = (\mathbf{B}, h)$ is $\mathscr{A}$-special. $\square$

We may give a characterization of $\mathscr{A}$-special structures directly in terms of a collection axiom.

Theorem 3.14. *Let $\mathscr{A} = (\mathbf{A}, R)$ be $+$-admissible. Let $\mathscr{C} = (\mathbf{C}, S)$. Then the following are equivalent:*
(A) *$\mathscr{C}$ is $\mathscr{A}$-special.*
(B) *$C \subseteq A$, C is transitive, closed under the taking of pairs and unions of families of sets, and for all X Δ_0 on $\mathbf{C}$ and $b \in C$, if*

$$(1) \qquad \forall x \in b\, \exists y_0, y_1 \in C\, \exists z_0, z_1 \in A[y_1 \subseteq S \wedge z_1 \subseteq R \wedge Xxy_0y_1z_0z_1]$$

then there are $c_0, c_1 \in C$, and $a_0, a_1 \in A$ such that $c_1 \subseteq S$, $a_1 \subseteq R$ and

$$(2) \qquad \forall x \in b\, \exists y_0, y_1 \in c_0\, \exists z_0, z_1 \in a_0[y_1 \subseteq c_1 \wedge z_1 \subseteq a_1 \wedge Xxy_0y_1z_0z_1].$$

Proof. $(A) \to (B)$: Assume (A) and (1). From (1) we infer

$$\forall x \in b\, \exists y_0, y_1, z_0', z_1'[y_1 \subseteq S \wedge z_0' \subseteq A \wedge z_1' \subseteq R \wedge \exists z_0, z_1 \in z_0' \\ (z_1 \subseteq z_1' \wedge Xxy_0y_1z_0z_1)].$$

By $+$-admissibility of $(\mathscr{C}, \mathscr{A})$, there are $c_0, c_1, a_0', a_1' \in C$ such that $c_1 \subseteq S$, $a_0' \subseteq A$, $a_1' \subseteq R$ and

$$\forall x \in b\, \exists y_0, y_1, z_0', z_1 \in c_0[y_1 \subseteq c_1 \wedge z_0' \subseteq a_0' \wedge z_1 \subseteq a_1' \wedge \exists z_0, z_1 \in z_0' \\ (z_1 \subseteq z_1' \wedge Xxy_0y_1z_0z_1)].$$

C is $\mathscr{A}$-bounded so we may find $a_0, a_1 \in A$ such that $a_0' \subseteq a_0$ and $a_1' \subseteq a_1 \subseteq R$. Then

$$\forall x \in b\, \exists y_0, y_1, z_0', z_1 \in c_0[y_1 \subseteq c_1 \wedge z_1' \subseteq a_1 \wedge \exists z_0, z_1 \in a_0 \\ (z_1 \subseteq z_1' \wedge Xxy_0y_1z_0z_1)].$$

Hence

$$\forall x \in b\, \exists y_0, y_1 \in c_0\, \exists z_0, z_1 \in a_0[y_1 \subseteq c_1 \wedge z_1 \subseteq a_1 \wedge Xxy_0y_1z_0z_1]$$

as required.
$(B) \to (A)$: Assume (B). (B) clearly implies that $(\mathscr{C}, \mathscr{A})$ is $+$-admissible. We show that C is $\mathscr{A}$-bounded. Let X be Σ^+ on $\mathscr{A}$ so there exists X' Δ_0 on $\mathbf{A}$ such that

$$Xx \leftrightarrow \exists z_0, z_1 \in A[z_1 \subseteq R \wedge X'xz_0z_1].$$

Now assume $c \in C$ and $c \subseteq X$. Then

$$\forall x \in c\, \exists z_0', z_1 \in A[z_1 \subseteq R \wedge (\exists z_0, y \in z_0')(y = x \wedge Xyz_0z_1)].$$

Applying (B) we get $a_0, a_1 \in A$ such that $a_1 \subseteq R$ and

$$\text{(3)} \qquad \forall x \in c\, \exists z_0', z_1 \in a_0[z_1 \subseteq a_1 \wedge (\exists z_0, y \in z_0')(y = x \wedge Xyz_0z_1)].$$

Let $a = \mathrm{TC}(a_0)$. Then $a \in A$.
Let

$$d = \{y \in a : (\exists z_0, z_1 \in a)(Xyz_0z_1 \wedge z_1 \subseteq a_1)\}.$$

By Δ_0-separation, $d \in A$. Also $c \subseteq d \subseteq X$ by definition of d and (3), completing the proof. □

Theorem 3.14 shows that $\mathscr{A}$-special structures are characterised by an appropriate generalisation of the set S of collection axioms in [16].

4. $\mathscr{A}$-Saturation and $\mathscr{A}$-special sets

Let A be PR-closed and let $\mathscr{A} = (\mathbf{A}, R_1, \ldots, R_n)$. Let K be an alphabet included in A.

Definition 4.1. A K-structure $\mathcal{M}$ is $\mathscr{A}$-saturated if it satisfies the following condition for every $I \in A$ and Γ Σ^+ on $\mathscr{A}$: suppose Θ is a function Σ^+ on $\mathscr{A}$, and for every $i \in I$ there is n_i such that $\theta_{ia} = \Theta(i, a)$ is an n_i-formula for all $a \in \Gamma$; then if $s \in M^n$, and

$$\mathfrak{M} \models \Big(\bigwedge\!\!\!\bigwedge_{i \in I} \forall v^i \bigvee\!\!\!\bigvee_{a \in \Gamma} \theta_{ia} \Big)[s]$$

where v^i is a sequence of the variables v_j for $n \leq j < n_i$, there is $\Delta \in A$ such that $\Delta \subseteq \Gamma$ and

$$\mathfrak{M} \models \Big(\bigwedge\!\!\!\bigwedge_{i \in I} \forall v^i \bigvee\!\!\!\bigvee_{a \in \Delta} \theta_{ia} \Big)[s].$$

Remark. It is not hard to see that this definition is equivalent to the notion obtained by putting Σ^+ for Σ in Definition 2.1 of [16]. It is a generalisation of the notion of recursively saturated structure (see [3]). We use "$\mathcal{A}$-saturated" rather than "Σ^+-saturated" because the $+$-admissible structure is not fixed throughout our discussion (as it was in [16]).

The following is a slight generalisation of Theorem 3.4(b) of [16].

Theorem 4.1. *Let $\mathcal{A} = (\mathbf{A}, R_1, \ldots, R_n)$, let K be an alphabet Δ^+ on $\mathcal{A}$ and suppose A is closed under $\mathrm{PR}(\chi_K)$ functions. Let $\mathfrak{M} = (M, \rho)$ be a K-structure with M a set of urelements disjoint from A. Then the following are equivalent:*

(i) *$\mathfrak{M}$ is $\mathcal{A}$-saturated*

(ii) *$(\mathbf{C}, \rho)$ is $\mathcal{A}$-special, where $C = A(\mathfrak{M})$*

(iii) *there is $\mathfrak{M}' = (M', \rho')$ isomorphic to $\mathfrak{M}$ and a set C such that $M' \in C$ and $(\mathbf{C}, \rho')$ is $\mathcal{A}$-special.*

Proof. (ii) $\to$ (iii) is clear and (iii) $\to$ (i) is essentially in Ressayre, for by 3.14, if $(\mathbf{C}, \rho')$ is $\mathcal{A}$-special, it satisfies the generalisation of the collection axioms S of [16] appropriate to this case.

We sketch a proof of (i) $\to$ (ii). Let $C = A(\mathfrak{M})$, and suppose $\mathfrak{M}$ is $\mathcal{A}$-saturated. Note that by 2.1, $\rho \subseteq C$.

We show first that C is $\mathcal{A}$-bounded. Suppose $c \in C$, Γ is Σ^+ on $\mathcal{A}$, and $c \subseteq \Gamma$. Now $c = D(\mathfrak{M}, a)(s)$, for some $a \in A$, $s \in M^{L(a)}$. Let F_1 be as in 2.2.

Then

$$(*) \qquad \mathfrak{M} \models \bigwedge_{y \in \operatorname{dom} a} \forall v^1 \bigvee_{b \in \Gamma} (a(y) \to F_1(y, b))[s]$$

where v^1 is a sequence of the variables v_{1i} for $i \in L(y)$. Applying Definition 4.1, there is $\Delta \in A$ such that $\Delta \subseteq \Gamma$ and (*) is true with Δ in place of Γ. But this means that $c \subseteq \Delta$, as required.

It remains to check the $+$-admissibility of $(\mathbf{C}, \rho)$. We shall leave the reader to construct most of this argument, quite similar to that immediately above and in Ressayre. From the hypothesis of a $+$-collection axiom follows an assertion like (*). Then the existence of a Δ as for (*) yields the conclusion of the $+$-collection axiom, the passage in both directions depending on 2.6.

The following observation is useful: let $c \in C$, $c \subseteq \rho$. Then $\operatorname{dom} c \in C$ and $\operatorname{dom} c \subseteq A$ so there is $\Delta \in A$ such that $\operatorname{dom} c \subseteq \Delta$ and hence $c \subseteq \rho \restriction \Delta$; hence a formula "$\exists y \subseteq \rho \cdots y \cdots$" may be replaced by "$\exists \Delta \in A \cdots \rho \restriction \Delta \cdots$". (To apply this, "$\cdots y \cdots$" must be increasing in y.) $\square$

We now state a slight generalisation of an important theorem of Ressayre.

Theorem 4.2. *Let $\mathscr{A}$ be countable $+$-admissible, and T a set of sentences Σ^+ on $\mathscr{A}$. If T has a model, T has an $\mathscr{A}$-saturated model.*

Proof. We apply Proposition 1.3 and merely sketch the proof. In this case we take $\mathscr{F}$ to be the collection of formulas which are conjunctions of Σ^+ subcollections of $\mathscr{A}$. We take as our γ_n's the sentences of the form:

$$\forall u\left[\left(\bigvee_{i\in I}\exists v^i \bigwedge_{\alpha\in\Gamma}\theta_{i\alpha}\right)\vee\bigvee_{\substack{\Delta\in A\\ \Delta\subseteq\Gamma}}\left(\neg\bigvee_{i\in I}\exists v^i\bigwedge_{\alpha\in\Delta}\theta_{i\alpha}\right)\right]$$

(see Definition 4.1 for the conditions on I, Γ, $\theta_{i\alpha}$). Verification of the hypotheses of 1.3 is almost a routine application of Barwise Compactness (3.10). $\square$

Ressayre proved the above theorem (for Σ, not Σ^+) from the following, apparently much stronger, theorem: let $K\subseteq K'$ be alphabets Δ on a countable admissible set A; suppose that $\mathfrak{M}$ is an $\mathbf{A}$-saturated K-structure, and that T' is a collection of K'-sentences Σ on $\mathbf{A}$ with the property that whenever $T'\vdash\varphi$ for $\varphi\in\mathscr{L}_{\infty_0}(K)\cap A$, $\mathfrak{M}\models\varphi$; then there is an $\mathbf{A}$-saturated K'-structure $\mathfrak{M}'$ such that $\mathfrak{M}'\restriction K=\mathfrak{M}$ and $\mathfrak{M}'$ is a model of T'. He proved it by a direct argument which, while reminiscent of an omitting-types theorem, is apparently not directly reducible to 1.3. In fact, this stronger theorem (for Σ^+, not just Σ) is a fairly direct consequence of 4.2 using results accumulated so far. We shall now give a short sketch of the proof for a countable admissible set A.

Assume all the hypotheses and let $\mathfrak{M}=(M,\rho)$, M a set of urelements disjoint from A, and let $C=A(\mathfrak{M})$. By 4.1, (C,ρ) is $\mathbf{A}$-special. Let

$$K_1=K'\cup\{\hat{m}:m\in M\}$$

and let

$$T_1=T'\cup\{\theta^{\mathfrak{M}}_{K_0}:K_0\in A,\,K_0\subseteq K\}$$

where $\theta^{M}_{K_0}$ is the conjunction of

$$\forall v_0\bigvee_{m\in M}v_0=\hat{m}$$

and the atomic diagram of $\mathfrak{M}\restriction K_0$. There is a name a_{K_0} of A such that $D(\mathfrak{M}',a_{K_0})=\theta^{\mathfrak{M}'}_{K_0}$ for all $\mathfrak{M}'$. If T_1 has a model then by 4.2 it has a $(\mathbf{C},\rho,A)$-saturated model, which is $\mathbf{A}$-saturated by 4.1 and 3.13. Further, it is a fortiori a model of T', and its reduct to K is $\mathfrak{M}$. If T_1 has no model, then by 3.10, and the $\mathbf{A}$-boundedness of C, there are $\Delta\in A$ and $K_0\in A$ such that

$$\vdash\theta^{\mathfrak{M}}_{K_0}\to\neg\bigwedge\Delta,\quad\text{and}\quad\Delta\subseteq T'.$$

Hence

$$\boldsymbol{C} \models (\exists v_1 \eta)[\theta^{\mathfrak{M}}_{K_0} \to \neg \bigwedge\!\!\!\bigwedge \Delta],$$

where $\Pr(v_0)$ (cf. 1.1) is $\exists v\eta$ and η is Δ_0. So there is $c \in C$ such that

$$\boldsymbol{C} \models \eta[\theta^{\mathfrak{M}}_{K_0} \to \neg \bigwedge\!\!\!\bigwedge \Delta, c].$$

Let φ be the sentence obtained by applying 2.6 to η and names for c and $\theta^{\mathfrak{M}}_{K_0} \to \neg \bigwedge\!\!\!\bigwedge \Delta$ (this latter name, by a remark above, picked uniformly for all $\mathfrak{M}$) and existentially quantifying over all free variables. Then $\mathfrak{M} \models \varphi$ and $\varphi \in \mathscr{L}_{\infty\omega}(K) \cap A$. Further, if $\mathfrak{M}'$ is a K-structure and $\mathfrak{M}' \models \varphi$, then by the choice of φ,

$$A(\mathfrak{M}') \models \Pr[\theta^{\mathfrak{M}'}_{K_0} \to \neg \bigwedge\!\!\!\bigwedge \Delta];$$

now clearly $\mathfrak{M}' \models \theta^{\mathfrak{M}'}_{K_0}$, so $\mathfrak{M}' \models \neg \bigwedge\!\!\!\bigwedge \Delta$. Hence $T \vdash \neg\varphi$, contradicting the hypotheses.

It should be noted that to prove this theorem even merely in the case of an admissible set A, 4.2 is needed for $+$-admissible structures.

A further useful connection between $\mathscr{A}$-saturation and $\mathscr{A}$-special structures is the following:

Theorem 4.3. *Let $\mathscr{A} = (\mathbf{A}, R_1, \ldots, R_n)$ be $+$-admissible. Let K be an alphabet, Δ^+ on $\mathscr{A}$, containing ε, U, and the constants $\hat{a}$ for $a \in A$, and let $K_0 = \{\varepsilon, \mathrm{U}\}$. Let $\mathfrak{N}$ be an $\mathscr{A}$-saturated K-structure for which $\hat{a}^{\mathfrak{N}} = a$ for all $a \in A$. Suppose $\mathfrak{N} \upharpoonright K_0$ is a normal endextension of A and a model of* KPU. *Then $(\mathfrak{N} \upharpoonright K_0)^{\mathrm{WF}}$ is $\mathscr{A}$-special.*

Proof. For simplicity we assume $n = 1$, $R_1 = R$. Let $C = WF(\mathfrak{N} \upharpoonright K_0)$. We first show that C is $\mathscr{A}$-bounded.

Let Γ be Σ^+ on $\mathscr{A}$ and let $c \in C$ be included in Γ. Then

$$\mathfrak{N} \models \forall v_1 \Big(\bigvee\!\!\!\bigvee_{a \in \Gamma} v_1 = \hat{a} \vee \neg v_1 \varepsilon v_0 \Big)[c].$$

By the $\mathscr{A}$-saturation of $\mathfrak{N}$, there is $\Delta \in A$ such that $\Delta \subseteq \Gamma$ and

$$\mathfrak{N} \models \forall v_1 \Big(\bigvee\!\!\!\bigvee_{a \in \Delta} v_1 = \hat{a} \vee \neg v_1 \varepsilon v_0 \Big)[c].$$

Hence $c \subseteq \Delta$, as required.

It follows by 3.12(i) that C and A have the same ordinals.

We now show that $(\boldsymbol{C}, A, R)$ is $+$-admissible. Suppose $c \in C$, $Xxy_0y_1y_2$ is Δ_0 on $\boldsymbol{C}$ and increasing in y_0, y_1, y_2, and

$$\forall x \in c\, \exists y_0, y_1, y_2 \in C[y_1 \subseteq A \wedge y_2 \subseteq R \wedge Xxy_0y_1y_2].$$

By the $\mathscr{A}$-boundedness of C,

$$\forall x \in c\, \exists y_0 \in Cy_1, y_2 \in A[y_2 \subseteq R \wedge Xxy_0y_1y_2].$$

Since, if $y_0 \in C$, then $\operatorname{rk}(y_0) \in A$,

$$\forall x \in c\, \exists y_0, u, f \in C\, \exists \beta, y_1, y_2 \in A[y_2 \subseteq R \wedge \operatorname{Trans}(u) \wedge y_0 \in u \wedge \text{“}f = \operatorname{rk} \upharpoonright u\text{”} \wedge f : u \to \beta \wedge Xxy_0y_1y_2].$$

We may assume that K has a name $\dot{c}$ for c, as well as names for parameters needed in a Δ_0 definition of X, since a structure obtained by adding finitely many names for elements of the universe to an $\mathscr{A}$-saturated structure is $\mathscr{A}$-saturated.

For $\beta, a_1, a_2 \in A$, let $\theta_{\langle \beta, a_1, a_2\rangle}(v_0)$ be the formula

$$(1) \qquad \exists v_1, v_2, v_3[\text{“}\operatorname{Trans}(v_2) \wedge v_3 : v_2 \to \dot{\beta} \wedge v_3 = \operatorname{rk} \upharpoonright v_2\text{”} \wedge v_1 \in v_2 \wedge Xv_0v_1\dot{a}_1\dot{a}_2]$$

where by X in (1) we mean some Δ_0-formula defining it.

We claim that

$$\mathfrak{N} \models \forall v_0 (\neg v_0 \varepsilon \dot{c} \vee \bigvee\!\!\!\bigvee_{\substack{\beta, a_1, a_2 \in A \\ a_2 \subseteq R}} \theta_{\langle \beta, a_1, a_2\rangle}(v_0)).$$

Now we justify the claim. Any element of N satisfying $v_0 \varepsilon \dot{c}$ is in C. Further if $c' \in C$,

$$\mathfrak{N} \models \theta_{\langle \beta, a_1, a_2\rangle}[c'] \quad \text{if and only if} \quad \mathbf{C} \models \theta_{\langle \beta, a_1, a_2\rangle}[c'].$$

because any witnesses in $\mathfrak{N}$ for the unbounded existential quantifiers in (1) must all be in C, as their ranks must be in A, and because the matrix of (1) is Δ_0 and so absolute for endextensions.

As $\mathfrak{N}$ is $\mathscr{A}$-saturated, there are $\beta, a_1, a_2 \in A$ such that $a_2 \subseteq R$ and

$$\mathfrak{N} \models \forall v_0 (\neg v_0 \varepsilon \dot{c} \vee \theta_{\langle \beta, a_1, a_2\rangle}(v_0)).$$

(We have used the fact that X is increasing in y_0, y_1, and y_2 to reduce an A-finite set of β, a_1, a_2's to one.)

Combining this with all the remarks above, we conclude that

$$\forall x \in c\, \exists y_0 \in C \quad Xxya_1a_2$$

Now by the Barwise Truncation Lemma, C is admissible, and we may therefore bound y_0 in C, completing the proof. $\square$

If A is pure, each $a \in A$ is definable in any endextension of $\mathbf{A}$ by a formula φ_a of $\mathscr{L}_{\infty\omega}(\varepsilon) \cap A$ in such a way that $\{\langle a, \varphi_a\rangle : a \in A\}$ is Δ on $\mathbf{A}$. Hence we may weaken the hypotheses above and obtain:

Theorem 4.4. *Let $\mathscr{A} = (\mathbf{A}, R_1, \ldots, R_n)$ be $+$-admissible and let $\mathbf{A}$ be pure. If $\mathfrak{N}$ is an $\mathscr{A}$-saturated normal model of* KPU *which is an endextension of $\mathbf{A}$ then $\mathfrak{N}^{\mathrm{WF}}$ is $\mathscr{A}$-special.*

Theorem 4.3 is a useful device for obtaining $\mathcal{A}$-special sets, in much the same way as fattenings may be obtained by applying a theorem of Friedman [7].

We confine ourselves for the moment to pure sets. Friedman's theorem may be stated as follows: let A be countable admissible and let T be a collection of sentences Σ on $\mathbf{A}$, including KP, and having a model which is an endextension of $\mathbf{A}$; then T has a model which is an endextension of $\mathbf{A}$ whose standard ordinals are exactly those of A.

The well-founded part of such a model will extend A and have the same ordinals, i.e. it will be a fattening of A just as the well-founded part of an A-saturated model $\mathfrak{N}$ will be $\mathbf{A}$-special.

The limitation of the fattening is that, having it, no means are at hand to go further. In a sense, while we have gained an admissible set, we have lost the old one. With B $\mathbf{A}$-special, however, we have $(\mathbf{B}, A)+$-admissible; what was Σ on A is Σ^+ on $(\mathbf{B}, A)$, and any collection B can do inside A can be bounded by one in A. So the old "recursion theory" is not lost.

We shall now use 4.3 to analyse the relations between $A(\mathfrak{M})$ as defined in Section 2 and certain "pure" extensions of $\mathcal{A}$.

Theorem 4.5. *Let A be countable and let $\omega \in A$. Suppose that $\mathcal{A} = (\mathbf{A}, R_1, \ldots, R_n)$ is $+$-admissible, K is an alphabet Δ^+ on $\mathcal{A}$, and $\mathfrak{M} = \langle M, \rho \rangle$ is a countable $\mathcal{A}$-saturated K-structure, where M is a set of urelements disjoint from A. Then $A(\mathfrak{M}) \cap V_{U(A)}$ is the intersection of all sets C in $V_{U(A)}$ for which there is an isomorph $\mathfrak{M}' = \langle M', \rho' \rangle$ of $\mathfrak{M}$ such that $(\mathbf{C}, \rho')$ is $\mathcal{A}$-special.*

Proof. We shall assume further that M is infinite. The reader is left to see that the case of finite M is easily handled.

Let

$$D = A(\mathfrak{M}) \cap V_{U(A)}.$$

If $(\mathbf{C}, \rho')$ is $\mathcal{A}$-special and $\mathfrak{M}' = \langle M', \rho' \rangle$ is isomorphic to $\mathfrak{M}$, then C is PR-closed, includes A, and contains $\delta_\beta^{\mathfrak{M}'_0} = \delta_\beta^{\mathfrak{M}_0}$ for each $\beta \in A$ and $\mathfrak{M}'_0 = \mathfrak{M}' \upharpoonright K_0$ for each $K_0 \in A$ such that $K_0 \subseteq K$. Hence $D \subseteq C$ by 2.8. So D is included in the described intersection.

Suppose now that $Y \in V_{U(A)} - D$. It is enough to apply the following Lemma.

Lemma 4.6. *Let $\mathcal{A}, K, M$ be as above. Then if $Y \in V_{U(A)} - A(\mathfrak{M})$, there is a countable $C \in V_{U(A)}$ and an isomorph $\mathfrak{M}' = \langle M', \rho' \rangle$ of $\mathfrak{M}$ such that $(\mathbf{C}, \rho')$ is $\mathcal{A}$-special and $Y \in C$.*

Proof. We prove the lemma by induction on the rank of Y. So fix Y and assume the lemma true for each element of Y. It is enough to verify the lemma for Y.

We may clearly assume that $Y \subseteq A(\mathfrak{M})$. Otherwise, pick $Y' \in Y - A(\mathfrak{M})$ and apply the theorem to Y', by the induction hypothesis. Then the C and $\mathfrak{M}'$ so obtained work also for Y, as C is transitive.

Let $\mathfrak{B} = (A(\mathfrak{M}), \rho, A)$. By 4.1, $\mathfrak{B}$ is $+$-admissible. Much as in 4.2, we shall use the omitting types theorem 1.3. Let $U' = U(A) \cup M$.

Let K' be the alphabet containing ε, U, $\hat{a}$ for each $a \in A(\mathfrak{M})$, and a constant symbol F. Let T be the set of the following sentences of $\mathscr{L}_{\omega_1\omega}(K')$:

(i) KPU,

(ii) U$\hat{x}$ for $x \in U'$,

$$\forall v_0(v_0 \varepsilon \hat{a} \leftrightarrow \bigvee\!\!\!\bigvee_{b \in a} v_0 = \hat{b}) \quad \text{for sets } a \in A(\mathfrak{M}),$$

and

(iii) "F is a $1-1$ map of M onto $\hat{\omega}$."

Observe that T' is Σ^+ on $\mathfrak{B}$. Also T' has a model $\mathcal{N}$, with universe $HC \cap V_{U'}$, $\varepsilon^{\mathfrak{N}} = \in \restriction |\mathfrak{N}|^2$, $U^{\mathfrak{N}} = U'$, and $F^{\mathfrak{N}}$ some (any) bijection between M and ω.

Let $\mathscr{F}$ be the fragment of $\mathscr{L}_{\infty\omega}(K')$ consisting of formulas which are conjunctions of sets of formulas Σ^+ on $\mathfrak{B}$. Let γ_n, $n \in \omega$, be the set of sentences used to assure that any model of all of them will be $\mathfrak{B}$-saturated (as in the proof of 4.2). Let σ_Y be the sentence

$$\forall v_0\left[\exists v_1 \varepsilon v_0\left(\bigwedge\!\!\!\bigwedge_{a \in A(\mathfrak{M})} \neg v_1 = \hat{a}\right) \vee \bigvee\!\!\!\bigvee_{a \in Y} \neg \hat{a} \varepsilon v_0 \vee \bigvee\!\!\!\bigvee_{a \in A(\mathfrak{M}) - Y} \hat{a} \varepsilon v_0\right]$$

Then σ_Y is of the form $\forall v_0 \psi_Y$ where ψ_Y is $\bigvee\!\!\!\bigvee \exists$ over $\mathscr{F}$.

We shall check the hypotheses of 1.3 for σ_Y. Let $\eta(v_0, \ldots, v_k) \in \mathscr{F}$ and suppose that

$$T' \bigcup \{\eta(d_0, \ldots, d_k)\}$$

has a model, where the d_i are new constants, i.e. that $T' \cup \{\exists v_0 \cdots v_k \eta\}$ has a model.

Let η be $\bigwedge\!\!\!\bigwedge \Phi$ where Φ is a set of $(k+1)$-formulas Σ^+ on $\mathfrak{B}$. Let d be a new constant symbol different from all the d_i.

Suppose

(1) $\quad T' \cup \{\eta(d_0, \ldots, d_k), d \in d_0\} \cup \{\neg d = \hat{a} : a \in A(\mathfrak{M})\}$

has no model. Then by 3.10, there is $\Delta \in A(\mathfrak{B})$ such that

(2) $\quad T' \cup \{\eta(d_0, \ldots, d_k), d \varepsilon d_0\} \cup \{\neg d = \hat{a} : a \in \Delta\}$

has no model.

Hence

(3) $\quad T' \cup \{\eta(d_0, \ldots, d_k)\} \vdash \forall v_0 (v_0 \varepsilon d_0 \rightarrow v_0 \varepsilon \hat{\Delta})$

by (2) and the sentences (ii) of T'.

We claim now that there must be an $a \in Y$ such that

$T' \cup \{\eta(d_0, \ldots, d_k), \neg \hat{a} \varepsilon d_0\}$ has a model, or an $a \in A(\mathfrak{M}) - Y$ such that $T' \cup \{\eta(d_0, \ldots, d_k), \hat{a} \varepsilon d_0\}$ has a model.

For otherwise $Y \subseteq \Delta$ and is the set of $a \in \Delta$ such that

$$T' \cup \{\eta(d_0, \ldots, d_k), \neg \hat{a} \varepsilon d_0\}$$

has no model, which is (by 3.10, 1.1) Σ^+ on $\mathscr{A}$. It is also the set of $a \in \Delta$ such that $T' \cup \{\eta(d_0, \ldots, d_k), \hat{a} \varepsilon d_0\}$ has a model — that is, $\Delta - Y$ is Σ^+ on $\mathscr{A}$. Hence Y is a Δ^+ subset of Δ and by 3.4, is an element of $A(\mathfrak{M})$, contradicting our hypotheses.

So the claim is established. If $a \in Y$ and $T' \cup \{\eta(d_0, \ldots, d_k), \hat{a} \varepsilon d_0\}$ has a model, then

$$T' \cup \{\exists v_0 \cdots v_k\, (\eta \wedge \psi_Y)\}$$

has a model. We obtain the same conclusion in the other case of the claim.

Further, if (1) has a model (the only case remaining) then again

$$T' \cup \{\eta(d_0, \ldots, d_k), \psi_Y(d_0)\}$$

has a model, so $T' \cup \{\exists v_0 \cdots v_k (\eta \wedge \psi_Y)\}$ has a model, as required.

Hence we may conclude, by 1.3, that T' has a model $\mathfrak{N}$ satisfying all the γ_n's and σ_Y. Since $\mathfrak{N} \models \bigwedge\!\!\!\bigwedge_n \gamma_n$, it is $\mathscr{B}$-saturated.

Let $C' = \mathrm{WF}(\mathfrak{N})$. We may assume that C' is a transitive set extending $A(\mathfrak{M})$ (by (ii) of T', that $\hat{a}^{\mathfrak{N}} = a$ for $a \in A(\mathfrak{M})$, and that $\boldsymbol{C}'$ is $\mathscr{B}$-special (by 4.3), and so $\mathscr{A}$-special, by 2.12. Let $f = F^{\mathfrak{N}}$. Then $f \in C'$, since (iii) of T' requires f to be a set of ordered pairs of elements of C'. Further, f is a 1–1 map of M onto ω. Let $\mathfrak{M}' = \langle \omega, \rho' \rangle$ be the structure isomorphic to $\mathfrak{M}$ by the isomorphism f.

Now ρ' is Σ^+ on $\langle \boldsymbol{C}', \mathscr{B} \rangle$ so by 3.8, $(\boldsymbol{C}', \rho', \mathscr{A})$ is $+$-admissible. Let

$$C = \bigcup \{L_\alpha(\{a, \rho' \restriction K_0\}) : a \in A,\ K_0 \in A,\ K_0 \subseteq K\}.$$

By 3.13, $(\boldsymbol{C}, \rho')$ is $\mathscr{A}$-special. Then it is easily verified that $(\boldsymbol{C}, \rho', \mathscr{A})$ is $+$-admissible.

We need only verify now that Y is not an element of C. If $Y \in C'$,

$$\mathfrak{N} \models \Big(\exists v_1 \varepsilon v_0 \Big(\bigwedge\!\!\!\bigwedge_{a \in A(\mathfrak{M})} \neg v_1 = \hat{a}\Big) \vee \bigvee\!\!\!\bigvee_{a \in Y} \neg \hat{a} \varepsilon v_0 \vee \bigvee\!\!\!\bigvee_{a \in A(\mathfrak{M}) - Y} \hat{a} \varepsilon v_0\Big)[Y].$$

Y has no element not in $A(\mathfrak{M})$. So either there is $a \in Y$ such that $\mathfrak{N} \models (\neg \hat{a} \varepsilon v_0)[Y]$ or there is $a \in A(\mathfrak{M}) - Y$ such that $\mathfrak{N} \models (\hat{a} \varepsilon v_0)[Y]$. In the first case, we obtain the contradiction $a = \hat{a}^{\mathfrak{N}} \in Y$. A similar contradiction arises in the second case.

Hence Y is not in C' and thus certainly not in C. Therefore C and $\mathfrak{M}'$ satisfy the conclusion of the lemma, and we are finished. □

In the case that $U(A) = 0$ we may conclude that the pure part of $A(\mathfrak{M})$ is the intersection of all pure sets C for which there exists an isomorph $\mathfrak{M}' = \langle \omega, \rho' \rangle$ of $\mathfrak{M}$

such that $(\boldsymbol{C}, \rho')$ is $\mathscr{A}$-special. If $K \in A$, it is then the intersection of all pure $\mathscr{A}$-special extensions of A containing an isomorph of $\mathfrak{M}$.

This last is a generalisation of a result of Nadel-Stavi [14]: the pure part of HYP $(\mathfrak{M})$ is the intersection of all pure admissible sets containing an isomorph of $\mathfrak{M}$ (assuming, of course, that $\omega \in \mathrm{HYP}(\mathfrak{M})$).

The reader should note that the hypotheses on A in 4.5 are stronger than necessary. It is enough that A contain some infinite set, not necessarily ω. As we admit sets with urelements, this observation is not vacuous.

In [16], Ressayre showed that if A is a pure admissible set, $K \in A$, $\mathfrak{M}$ is an $\mathscr{A}$-saturated structure such that $|\mathfrak{M}| \in A$ and all the elements of $\mathfrak{M}$ are named in $\mathfrak{M}$, then the PR-closure of $A \cup \{\mathfrak{M}\}$ is $\boldsymbol{A}$-special. This follows easily from 4.1 and 3.11 (iii).

Thus there has been no loss in passing to urelement versions of this theorems; in fact, a significant uniformity is obtained.

Consider now the following situation: $(\boldsymbol{C}, \rho)$ is $\boldsymbol{A}$-special, $\mathfrak{M} = \langle M, \rho \rangle$ is a structure and $M \in C$; then there is a least set B such that $M \in B$ and $(\boldsymbol{B}, \rho)$ is $\boldsymbol{A}$-special. It makes sense then to call B $A(\mathfrak{M})$. B can be obtained by applying 3.13 with $h = \rho \cup \{\langle 0, M \rangle\}$. A warning is in order, however; the existence of $A(\mathfrak{M})$ does not imply the existence of a similar (e.g. $\boldsymbol{A}$-special) $A(\mathfrak{M}')$ even when $\mathfrak{M}'$ is isomorphic to $\mathfrak{M}$. Even for an A-finite language, as long as the universe may be an arbitrary set, the existence of a least $\boldsymbol{A}$-special set containing $\mathfrak{M}$ (or even of a fattening containing $\mathfrak{M}$) depends on the existence of some $\boldsymbol{A}$-special extension containing $\mathfrak{M}$.

We take the opportunity to indicate another contrast with the notion of fattening. If C is a fattening of A and $x \in C$, it is far from clear that there is a least fattening of A, and hence least admissible extension of A, containing x. As far as the author knows, this question is open.

5. Forcing and $\boldsymbol{A}$-special structures

We now wish to make good on two promises made above. We show in this section that a set-Cohen-generic extension of an admissible set A is $\boldsymbol{A}$-special; We show also that there are A, B pure countable admissible, such that B is a fattening of A and not $\boldsymbol{A}$-special.

We shall be concerned throughout this section with the method of forcing, first devised and applied to models of set theory by Cohen [6] and later adapted by Barwise and others for use in admissible sets.

Let $\mathscr{A} = (\boldsymbol{A}, R_1, \ldots, R_n)$ be countable $+$-admissible. We begin by assuming we have a partial order $\mathscr{Q} = \langle Q, \leqslant \rangle$ such that $Q \subseteq A$. A subset D of Q is said to be dense open if

(i) $(\forall p, q \in Q)(p \in D \wedge q \leqslant p \rightarrow q \in D)$, and

(ii) $(\forall p \in Q)(\exists q \in Q)(q \geqslant p \wedge q \in D)$.

If $\mathcal{D}$ is a collection of dense open subsets of Q, $G \subseteq Q$ is $\mathcal{Q}$-$\mathcal{D}$-generic if

(i) $(\forall p \in Q)(\exists q \in Q)(q \in G \wedge p \leqslant q \rightarrow p \in G)$,
(ii) $(\forall p, q \in G)(\exists r \in G)(p \leqslant r \wedge q \leqslant r)$, and
(iii) $(\forall D \in \mathcal{D})(\exists q \in G)(q \in D)$.

We shall need the following standard result.

Proposition 5.1. *If Q and $\mathcal{D}$ are countable, $\mathcal{Q}$-$\mathcal{D}$-generic sets exist.*

We wish to construct certain extensions of A from sets G $\mathcal{Q}$-$\mathcal{D}$-generic for sufficiently large families $\mathcal{D}$. The standard procedure is to directly define a class of names in A for elements of the extension and then define a forcing relation between elements of Q and sentences of set theory involving these names. We have already available from Section 2 a system of names and a reduction of certain sentences of set theory involving them to sentences of $\mathcal{L}_{\infty\omega} \cap A$ on some structure. We shall adapt these materials below.

Suppose henceforth that Q and $\leqslant$ are Δ^+ on A. Let $K_Q = \{\dot{q} : q \in Q\}$ where $\dot{q}$ is the 0-ary relation symbol $\langle 0, \langle 0, \langle 0, q\rangle\rangle\rangle$ and, for $G \subseteq Q$, let $A_G = A(\mathfrak{M}_G)$ where $\mathfrak{M}_G = \langle 0, \rho_G\rangle$ when

$$\rho_G(\dot{q}) = \begin{cases} 0 & \text{if} \quad q \in G \\ 1 & \text{if} \quad q \notin G \end{cases}$$

That is, $\mathfrak{M}_G$ is the empty structure with a family of nullary relations. Now by Proposition 2.4, A_G is PR-closed. Furthe:, we do not need all of N to name elements of A_G. Let N' be defined recursively as follows:

$$x \in N' \quad \text{if} \quad x \in N \wedge L(x) = 0 \wedge (\forall y \in \operatorname{dom} x)(y \in N').$$

Proposition 5.2. *There is a function H_1 which is primitive recursive, and satisfies*

(i) *if $x \in N$, $H_1(x) \in N'$, and*
(ii) *for all $\mathfrak{M}$ with universe 0, $D(\mathfrak{M}, x) = D(\mathfrak{M}, H_1(x))$.*

Proof. H_1 is defined recursively by setting $H_1(x) = 0$ is $L(x) \neq 0$ and

$$H_1(x) = \{\langle H_1(y), x(y)\rangle : L(y) = 0 \quad \text{and} \quad y \in \operatorname{dom} x\}.$$

otherwise. $\square$

By Proposition 2.6 there is a PR funct'on F_1 such that if φ is a Δ_0 n-formula of $L(\varepsilon, U)$ and $x_0, \ldots, x_{n-1} \in N$, then for all G,

$$(1) \qquad \mathbf{A}_G \vDash \varphi[D(\mathfrak{M}_G, x_1), \ldots, D(\mathfrak{M}_G, x_n)] \leftrightarrow \mathfrak{M}_G \vDash F_1(\varphi, \langle x_0 \cdots x_{n-1}\rangle)$$

where we write $D(\mathfrak{M}, x)$ for $D(\mathfrak{M}, x)(0)$, as $L(x) = 0$ for all $x \in N$ in which we are interested in this section. If $x_0, \ldots, x_{n-1}$ are in N', we see easily that

$F_1(\varphi, \langle x_0, \ldots, x_{n-1}\rangle)$ is in VF $\mathscr{L}_{\infty\omega}$, the class of formulas of $\mathscr{L}_{\infty\omega}$ in which no variables occur. Furthermore, since K_Q is $\Delta^{\mathscr{A}}$ on $\mathscr{A}$ we may get F which is PR(χ_Q), satisfies (1) above for all $\varphi, x_0, \ldots, x_{n-1}$, and also such that $F(\varphi, \langle x_0 \cdots x_{n-1}\rangle) \in$ VF $\mathscr{L}_{\infty\omega}(K_Q)$ when $x_0, \ldots, x_{n-1} \in N^r$.

It follows that if ψ is, say, $\forall v_1 \exists v_2 \cdots \exists v_n \varphi$, where φ is Δ_0, then

$$\mathbf{A}_G \models \psi \leftrightarrow \mathfrak{M}_G \models \bigwedge\!\!\!\bigwedge_{a_1 \in A} \bigvee\!\!\!\bigvee_{a_2 \in A} \cdots \bigvee\!\!\!\bigvee_{a_n \in A} F(\varphi, \langle a_1, \ldots, a_n\rangle).$$

This observation is used in 5.5 below.

It is between elements of Q and sentences of the form above that we need the forcing relation.

Definition 5.1. For $\varphi \in$ VF $\mathscr{L}_{\infty\omega}(K_Q)$ and $p \in Q$ we define, by induction on φ, the notion "$p \Vdash_{\mathscr{Q}} \varphi$" as follows:

(i) $p \Vdash_{\mathscr{Q}} \dot{q}$ if $q \leqslant p$ in $\mathscr{Q}$

(ii) $p \Vdash_{\mathscr{Q}} \neg\varphi$ if there is no $q \geqslant p$ such that $q \Vdash_{\mathscr{Q}} \varphi$

(iii) $p \Vdash_{\mathscr{Q}} \bigvee\!\!\!\bigvee \Phi$ if there is $\varphi \in \Phi$ such that $p \Vdash_{\mathscr{Q}} \varphi$.

We shall often drop the subscript $\mathscr{Q}$ if its value is clear from the context or is temporarily fixed.

Lemma 5.3. (i) *If $p \leqslant q$ and $p \Vdash \varphi$, then $q \Vdash \varphi$.*

(ii) *for all $p \in Q$, there is $q \in Q$ such that $q \Vdash \varphi$ or $q \Vdash \neg\varphi$, and $p \leqslant q$.*

Lemma 5.4. *Let G be $\mathscr{Q}$-$\mathscr{D}$-generic. Suppose that for every subformula ψ of $\varphi \in$ VF $\mathscr{L}_{\infty\omega}(K_Q)$, $\{p \in Q : p \Vdash \psi \vee p \Vdash \neg\psi\} \in \mathscr{D}$. Then*

$$\mathfrak{M}_G \models \varphi \leftrightarrow (\exists p \in G)(p \Vdash \varphi).$$

Proof. A simple induction on φ.

The point of 5.4 above is made clearer by:

Proposition 5.5. (i) *Suppose $\mathscr{Q} \in A$. Then*

$$\{\langle p, \varphi\rangle : p \in A, \varphi \in \text{VF}\, \mathscr{L}_{\infty\omega}(K_Q) \cap A, p \Vdash \varphi\}$$

is Σ on A.

(ii) *If $F : A^n \to A$ is $\Delta^{\mathscr{A}}$ on $\mathscr{A}$ and for all $a_1, \ldots, a_n \in A$, $F(a_1, \ldots, a_n) \in$ VF $\mathscr{L}_{\infty\omega}(K_Q)$, then*

$$\{p : p \Vdash \bigwedge\!\!\!\bigwedge_{a_1 \in A} \cdots \bigvee\!\!\!\bigvee_{a_n \in A} F(a_1, \ldots, a_n)\}$$

is definable on $\mathscr{A}$

Proof. (i) The relation mentioned is in fact PR in Q.
(ii) Induction on n. $\square$

We are now prepared to prove one of the facts asserted above:

Theorem 5.6. *Suppose $\mathfrak{Q} \in A$, $\mathfrak{D}$ = all dense open subsets of Q definable on $\mathscr{A}$. Let G be $\mathfrak{Q}$-$\mathfrak{D}$-generic. Then $\mathbf{A}_G$ is $\mathscr{A}$-special.*

Proof. We apply 4.1. Note that 0 is a set of urelements disjoint from A.
Suppose F is PR, Γ is Σ^+ on $\mathscr{A}$, $I \in A$ and

$$\mathfrak{M}_G \models \bigwedge\!\!\!\bigwedge_{i \in I} \bigvee\!\!\!\bigvee_{a \in \Gamma} F(i, a)$$

Then by 5.5 and 5.4, there is $p \in G$ such that

$$p \Vdash \bigwedge\!\!\!\bigwedge_{i \in I} \bigvee\!\!\!\bigvee_{a \in \Gamma} F(i, a).$$

Unraveling the definition,

$$\forall i \in I\, \forall q \in Q(q \geqslant p \to (\exists r \in Q)(r \geqslant q \wedge (\exists a)(a \in \Gamma \wedge r \Vdash F(i, a)))$$

That is,

$$\forall \langle i, q\rangle \in I \times Q\, \exists a(a \in \Gamma \wedge (\exists r \in Q)(q \geqslant p \to r \geqslant q \wedge r \Vdash F(i, a)))$$

By 5.5(i) the part in brackets is Σ on $\mathbf{A}$, so by 3.5, there is $\Delta \in A$ such that $\Delta \subseteq \Gamma$ and

$$\forall \langle i, q\rangle \in I \times Q\, \exists a(a \in \Delta \wedge (\exists r \in Q)(q \geqslant p \to r \geqslant q \wedge r \Vdash F(i, a)))$$

But using the definition again, this means that

$$p \Vdash \bigwedge\!\!\!\bigwedge_{i \in I} \bigvee\!\!\!\bigvee_{a \in \Delta} F(i, a).$$

and so by 5.5 and 5.4,

$$\mathfrak{M}_G \models \bigwedge\!\!\!\bigwedge_{i \in I} \bigvee\!\!\!\bigvee_{a \in \Delta} F(i, a).$$

Therefore $\mathfrak{M}_G$ is $\mathscr{A}$-saturated, and so $\mathbf{A}_G$ is $\mathscr{A}$-special. $\square$

We should at least take note of the result we obtained along the way:

Proposition 5.7. *Assume the hypotheses of 5.6. Then $\mathfrak{M}_G$ is $\mathscr{A}$-saturated.*

When $\mathbf{A}$ also happens to be a model of ZF, a generic set G gives rise to a set A_G in which $\langle A_G, A\rangle$ is also a model of ZF, that is, in the language with a unary predicate symbol for A. (This increases the available applications of the axioms of replacement and separation.) Checking an instance of replacement in the extension involves an application of replacement in A.

When dealing only with admissible sets A, $(\mathbf{A}_G, A)$ is no longer fully admissible because an instance of Δ_0-collection in the extension may require for its truth an instance, say, of π_1-collection in $\mathbf{A}$. If A appears only positively, however, only instances of Σ_1-collection will be called for. Thus, as proven, $(\mathbf{A}_G, A)$ is $+$-admissible (and more).

This completes the part of this section in which we relate $\mathbf{A}$-special structures and forcing. It is not clear to the author to what extent forcing (except in a most trivially general sense), $\mathscr{A}$-saturation, and related notions are all simply distinct instances of "one vague principle" (Grilliot [8]). There seems to be no clear and useful reduction of any one to another. However in many applications one will serve as well as the other.

For the rest of this section we shall apply rather well-known facts about forcing to find A, B admissible such that B is a fattening of A but is not $\mathbf{A}$-special. As this material is available in the literature for the case of a model of ZF (see, in particular, [17]), and the application to admissible sets involves only the book-keeping necessary to keep track of assumptions actually used, we shall be spare with proof.

We say G is $\mathscr{Q}$-$\mathscr{A}$-generic if G is $\mathscr{Q}$-$\mathscr{D}$-generic where $\mathscr{D}$ is the set of dense open subsets of Q definable on $\mathscr{A}$. "$\mathscr{Q}$-A-generic" means "$\mathscr{Q}$-$\mathbf{A}$-generic".

Lemma 5.8. (i) *Let $\mathscr{Q} = \mathscr{Q}_1 \times \mathscr{Q}_2$, $\mathscr{Q}_1 \in A$. Then G is $\mathscr{Q}$ – A-generic if and only if there are G_1, G_2 such that $G = G_1 \times G_2$, G_1 is $\mathscr{Q}_1$-A-generic, and G_2 is $\mathscr{Q}_2$-$(\mathbf{A}_{G_1}, A)$-generic.*

(ii) *Let $\mathscr{Q} = \mathscr{Q}_1 \times \mathscr{Q}_2$, $\mathscr{Q}_1 \in A$. Let G_1 be $\mathscr{Q}_1$-A-generic, D a dense open subset of $\mathscr{L}$ first-order definable on $\mathbf{A}$. Then there is $p \in G_1$, $q \in Q_2$ such that $\langle p, q\rangle \in D$.*

(iii) *Let $\mathscr{Q}_1, \mathscr{Q}_2 \in A$ and suppose there is $f \in A$ which is an isomorphism between $\mathscr{Q}_1$ and $\mathscr{Q}_2$. If G is $\mathscr{Q}_1$-A-generic, then $\{f(p) : p \in G\}$ is $\mathscr{Q}_2$-A-generic.*

Proof. Omitted. $\square$

Suppose now that α is the ordinal of A. Let

$$Q = \{f : (\exists X \text{ finite})(X \subseteq \alpha \times \omega \wedge f : x \to 2)\}.$$

Say $f_1 \leq f_2$ if $f_1 \subseteq f_2$, and let $\mathscr{Q} = \langle Q, \leq\rangle$. If $\beta < \alpha$, let

$$Q_\beta = \{f \in Q : \operatorname{dom} f \subseteq \beta \times \omega\},$$
$$Q^\beta = \{f \in Q : \operatorname{dom} f \subseteq (\alpha - \beta) \times \omega\},$$
$$\mathscr{Q}_\beta = \langle Q_\beta, \leq\rangle, \mathscr{Q}^\beta = \langle Q^\beta, \leq\rangle.$$

If $G \subseteq Q$, let $G_\beta = G \cap Q_\beta$. If $p \in Q$, $\beta < \alpha$, $p = p_\beta \cup p^\beta$ where

$$p_\beta = p \restriction (\beta \times \omega), \quad p^\beta = p \restriction (\alpha - \beta) \times \omega.$$

Using this we see that $\mathcal{Q} \approx \mathcal{Q}_\gamma \times \mathcal{Q}^\beta$.

Let "$\Vdash$" be $\Vdash_{\mathcal{Q}}$, "$\Vdash_\gamma$" be $\Vdash_{\mathcal{Q}_\gamma}$.

Lemma 5.9. *For $\varphi \in \mathrm{VF}\,\mathcal{L}_{\infty\omega}(K_Q)$, if $\mathrm{On} \cap \mathrm{TC}(\varphi) \subseteq \gamma$, then $p \Vdash \varphi \leftrightarrow p_\gamma \Vdash_\gamma \varphi$.*

Proof. We prove the theorem for all γ *by induction on* $\varphi \in \mathrm{VF}\,\mathcal{L}_{\infty\omega}(K_Q)$. It is trivial for atomic formulas, and the induction step for $\mathbb{W}$ is easy. We prove the induction step for $\neg$.

Let $\varphi = \neg\psi$, and assume the theorem for ψ and all γ. Suppose $\gamma \supseteq \mathrm{TC}(\varphi) \cap \mathrm{On} \supseteq \mathrm{TC}(\varphi) \cap \mathrm{On}$. If not $p_\gamma \Vdash_\gamma \varphi$ then there is $r \in Q_\gamma$ such that $p_\gamma \cup r \Vdash_\gamma \psi$. But then $p \cup r \in Q$, since p and r cannot clash on $\gamma \times \omega$ (because r and p_γ are compatible). Also $(p \cup r)_\gamma = p_\gamma \cup r \Vdash_\gamma \psi$ so by the induction hypothesis $p \cup r \Vdash \psi$. But then not $p \Vdash \varphi$.

If not $p \Vdash \varphi$ there is $r \in Q$ such that $p \cup r \Vdash \psi$. But then $(p \cup r)_\gamma = p_\gamma \cup r_\gamma \Vdash_\gamma \psi$ by the induction hypothesis, so not $p_\gamma \Vdash_\gamma \varphi$. Hence the induction step for $\neg$ is verified and the proof is complete. $\square$

Corollary 5.10. $\{\langle p, \varphi\rangle : p \in Q,\ \varphi \in \mathrm{VF}\,\mathcal{L}_{\infty\omega}(K) \cap A$, *and* $p \Vdash \varphi\}$ *is* Σ *on* $\mathbf{A}$.

We say $D \subseteq Q$ is dense open above p if $\{q \in Q : q \in D \text{ or } \neg q \geqslant p\}$ is dense open. Hence if D is dense open above p and $q \geqslant p$, there is $r \in D$ such that $r \geqslant q$.

We say that $\mathbf{A}$ satisfies Σ-DC (Σ Dependent Choices) if whenever X is a binary relation Σ on $\mathbf{A}$, $a \in A$, and

$$\forall x \in A\ \exists y \in A \quad Xxy$$

there is $f \in A$ such that $f : \omega \to A$, $f(0) = a$, and

$$\forall n \in \omega X f(n) f(n+1).$$

Lemma 5.11. *Let A be admissible and suppose $\mathbf{A}$ satisfies Σ-DC. Let $D \subseteq Q$ be dense open above p, and Σ on $\mathbf{A}$. Then there is $D' \in A$ such that $D' \subseteq D$ and $(\forall q \in Q)(q \geqslant p \to (\exists r \in D')\ (q \cup r \in D))$.*

Proof. We may (and do) assume that $q \in D \to q \geqslant p$.

Using Σ-DC we obtain a sequence $\langle X_n : n \in \omega\rangle$ of finite subsets of Q such that for all $n \in \omega$, if

$$Y_n = \left(\bigcup_{q \in X_n} \operatorname{dom} q\right) - \operatorname{dom} p,$$

and $f : Y_n \to 2$, there is $q \in X_{n+1} \cap D$ such that $q \supseteq p \cup f$. (We are using the fact that D is dense open above p.)

Let $D' = \bigcup_n X_n$. Then $D' \in A$. If $q \in Q$ and $q \geqslant p$, let

$$S = \operatorname{dom} q \cap \bigcup_n Y_n.$$

Then as q is finite, there is N such that $S \subseteq Y_N$. But then there is $r \in X_{N+1}$ such that $r \in D$ and $r \geqslant p \cup q \restriction S$. Then $q \cup r \in D$, as required. $\square$

Lemma 5.12. *Let A be countable admissible and suppose $\mathbf{A}$ satisfies Σ-DC. Let G be $\mathcal{Q}$-A-generic. Then $(\mathbf{A}_G, p_G)$ is $\mathbf{A}$-special.*

Proof. Assume the hypotheses of the theorem. Let Θ be PR, $I \in A$ and let Γ be Σ on $\mathbf{A}$. Suppose that if $i \in I$ and $a \in \Gamma$, $\theta_{ia} = \Theta(i, a)$ is a sentence of VF $\mathcal{L}_{\infty\omega}(K_Q)$.

If

$$\mathfrak{M}_G \models \bigwedge_{i \in I} \bigvee_{a \in \Gamma} \theta_{ia},$$

then by genericity of G, there is $p \in G$ such that

$$p \Vdash \bigwedge_{i \in I} \bigvee_{a \in \Gamma} \theta_{ia}.$$

That is,

$$\forall i \in I\, \exists q \in Q\, \exists a, r (a \in \Gamma \wedge (q \geqslant p \rightarrow r \geqslant q \wedge r \Vdash \Theta(i, a)). \qquad (*)$$

For $i \in I$, let

$$D_i = \{r \in Q : (\exists a \in A)(r \geqslant p \wedge r \Vdash \Theta(i, a))\}.$$

D_i is dense open above p, by (*), so by Lemma 5.11

$$\forall i \in I\, \exists D', \beta [D' \subseteq D_i \cap Q_\beta \wedge (\forall q \in Q_\beta)(q \geqslant p \rightarrow (\exists r \in D')(q \cup r \in D_i)]$$

Now if $D' \subseteq Q_\beta$, $(\exists r \in D')(q \cup r \in D_i)$ for all $q \in Q$ if and only if the same condition holds for all $q \in Q_\beta$.

Note that if $D' \subseteq D_i$ and $D' \in A$ there is $\Delta \in A$ such that

$$\Delta \subseteq \Gamma \quad \text{and} \quad \forall q \in D'\, \exists a \in \Delta (q \Vdash \theta(i, a)).$$

So

$$\forall i \in I\, \exists D', \beta, \Delta [D' \subseteq D_i \cap Q_\beta \wedge (\forall q \in Q_\beta)(q \geqslant p \rightarrow (\exists r \in D')(\exists a \in \Delta)(q \cup r \Vdash \Theta(i, a))].$$

So by Σ-collection in $\mathbf{A}$ there is Δ in A such that

$$\forall i \in I\, \exists D', \beta [D' \subseteq D_i \cap Q_\beta \wedge (\forall q \in Q_\beta)(q \geqslant p \rightarrow (\exists r \in D')(\exists a \in \Delta)(q \cup r \Vdash \Theta(i, a))].$$

By the remark above,

$$\forall i \in I\, \exists q \in Q(q \geq p \to (\exists r)(\exists a \in \Delta)(r \geq p \wedge r \Vdash \Theta(i, a))).$$

That is,

$$p \Vdash \bigwedge_{i \in I} \bigvee_{a \in \Delta} \theta(i, a).$$

Hence

$$\mathfrak{M}_G \models \bigwedge_{i \in I} \bigvee_{a \in \Delta} \theta(i, a).$$

By Theorem 4.1, $(\mathbf{A}_G, \rho_G)$ is admissible over $\mathbf{A}$. □

Theorem 5.13. *Let A be countable admissible, and suppose $\mathbf{A}$ satisfies Σ-DC. Suppose further that $\mathbf{A} \models$ "all sets are countable". Then there are countable fattenings B and C of A such that C is a fattening of B which is not $\mathbf{B}$-special. (Furthermore, B is Σ on C.)*

Proof. We shall obtain C by adjoining a sequence of subsets of ω to A. B will be a certain inner model of C.

Let Q' be the set of finite partial functions from $\omega \times \omega$ to 2, say $f_1 \leq f_2$ if $f_1 \subseteq f_2$ and let $\mathcal{Q}' = \langle Q', \leq \rangle$. Let G be $\mathcal{Q}'$-A-generic. Let $a = \bigcup G$ and for each n let

$$a_n = \{m \,.\, a(n, m) = 0\}$$

Let $C = A_G$.

If $m \in \omega$, let $Q'_m = \{f \in Q' : \operatorname{dom} f \subseteq m \times \omega\}$ and let $\mathcal{Q}'_m = \langle Q'_m, \subseteq \rangle$.

Let $\mathcal{Q}$ be the partial order of Theorem 5.12 and let Q_β, Q^β, p_β, p^β be as before. If G is $\mathcal{Q}$-A-generic, let $G_\beta = G \cap Q_\beta$ and $G^\beta = G \cap Q^\beta$.

Let $D_1, D_2, D_3, \ldots$ enumerate the dense open subsets of Q which are first order definable on $\mathbf{A}$. Let α_n, $n \in \omega$, be ordinals such that $\alpha_0 = 0$, $\alpha_n < \alpha_{n+1}$, and $\sup\{\alpha_n : n \in \omega\} = \alpha$, the ordinal of A.

We shall define, by induction on n, ordinals $\beta_n \in A$, and functions f_n and h_n in A, such that $\beta_n \geq \alpha_n$, f_n is a bijection from $(\beta_n - \beta_{n-1}) \times \omega$ to ω, $\{e : s \subseteq f_n, s \text{ finite}\}$ is $\mathcal{Q}_{\beta_n}$-A-generic, and h_n differs only finitely from

$$\bigcup_{m < n} (\chi_{a_m} \circ f_{m+1}).$$

Furthermore, if $n \geq 1$, there will be a $q \in D_n$ such that $q \subseteq h_n$.

Let $\beta_0 = h_0 = f_0 = 0$.

Suppose we have β_n, f_n, h_n satisfying all the conditions given. We now define β_{n+1}, h_{n+1}, and f_{n+1}. By 5.8 (ii) there are $q \in \mathcal{Q}^{\beta_n}$, $p \subseteq h_n$ such that $p \cup q \in D_n$. Choose $\beta_{n+1} > \alpha_{n+1} + \beta_n + 1$ so that $\operatorname{dom} q \subseteq \beta_{n+1} \times \omega$ and let f_{n+1} be chosen in A so that f_{n+1} is a bijection from $(\beta_{n+1} - \beta_n) \times \omega$ to ω. ($A \models$"all sets are countable".)

For $\gamma > \beta_{n+1}$ and $m \in \omega$, let

$$h_{n+1}(\gamma, m) = \begin{cases} h_n(\gamma, m) & \text{if } \gamma < \beta_n \\ q(\gamma, m) & \text{if } \langle \gamma, m\rangle \in \operatorname{dom} q \\ \chi_{a_n}(f_{n+1}(\gamma, m)) & \text{otherwise} \end{cases}$$

It is easy to check that β_{n+1}, h_{n+1}, and f_{n+1} have all the properties claimed.

Now $\{s : s \subseteq a \restriction ((n+1) \times \omega),\ s \text{ finite}\}$ is $\mathcal{Q}'_{n+1}$-A-generic, by 5.8 (i) and hence, using 5.8 (iii),

$$\{s \in Q_{\beta_{n+1}} : s \subseteq \bigcup_{m \leq n} (\chi_{a_m} \circ f_{m+1})$$

is $\mathcal{Q}_{\beta_{n+1}}$-$A$-generic. But as h_{n+1} differs only finitely from $\bigcup_{m \leq n} (\chi_{a_m} \circ f_{m+1})$, it is easy to see that $\{s \in Q_{\beta_{n+1}} : s \subseteq h_{n+1}\}$ is $\mathcal{Q}_{\beta_{n+1}}$-$A$-generic, so the construction may be continued.

Now let $h = \bigcup_n h_n$ and $G_h = \{p \in Q : p \subseteq h\}$. By construction, G_h meets every set D_n, and is therefore $\mathcal{Q}$-A-generic. Let $B = A_{G_h}$.

We know that B admissible by 5.12.

We claim that $B \subseteq C$. B is the PR-closure of

$$A \cup \{\langle 0, \rho_{G'}\rangle : G' = G \cap Q_\beta \text{ for some } \beta \in A\}.$$

From this it follows that B is the PR-closure of $A \cup \{h \restriction (\beta \times \omega) : \beta \in A\}$, and therefore, of $A \cup \{h_n : n \in \omega\}$. But each h_m differs only finitely from some element of C and is therefore itself in C. Since C is PR closed, $B \subseteq C$. It is also not hard to see that B, as the PR closure of

$$A \cup \{a \restriction (n \times \omega) : n \in \omega\}$$

is Σ on $\mathbf{C}$.

Let $c = \{a_n : n \in \omega\}$. $c \in C$ and $c \subseteq B$. Suppose $c \subseteq b \in B$. Then $b \in A_{G_n}$ for some $n \in \omega$, where $G_n = \{p \in G : \operatorname{dom} p \subseteq n \times \omega\}$. But by 5.8(i), G_n is $\mathcal{Q}_n$-A-generic for $Q_n = \{q \in Q : \operatorname{dom} q \subseteq n \times \omega\}$. A_{G_n} is transitive, so this implies that $a_n \in A_{G_n}$. But, by 5.8(i), a_n is $\mathcal{Q}^n$-A_{G_n}-generic (where $Q^n = \{q \in Q : \operatorname{dom} q \subseteq ((n+1) - n) \times \omega\}$). Since

$$\{q \in Q : (\exists x \in \omega)(q(n, x) = a_n(x))\}$$

is easily seen to be dense open in Q^n and definable on $\mathbf{A}_{G_n}$, we have a contradiction.

Hence C is not B-bounded, so, a fortiori, not admissible over $\mathbf{A}$, as required. $\square$

Corollary 5.14. *There is a pair B, C of countable admissible sets such that C is a fattening of B which is not $\mathbf{B}$-special.*

Proof. Note that L_{ω_1} satisfies the hypotheses of 5.13; thus, B and C may be chosen as fattenings of L_{ω_1}. (ω_1 is Church-Kleene ω_1, the first non-recursive ordinal.) □

Let us look at this example a little more closely. Let $\mathfrak{M}$ be the structure $\langle\omega, S, X\rangle$ where S is the successor function and X is the set of integers coding pairs m, n for which $a(\langle m, n\rangle)=0$. Then $\mathfrak{M}\in C$, which is a fattening of B. However, our argument above actually showed that no set containing $\mathfrak{M}$ and including A can be $\mathbf{B}$-special. Hence, $\mathfrak{M}$ cannot be $\mathbf{B}$-saturated.

We conclude that

Corollary 5.15. *There is a countable admissible set B and a structure $\mathfrak{M}$ such that $\mathfrak{M}$ is an element of a fattening of $\mathfrak{B}$ and $\mathfrak{M}$ is not $\mathbf{B}$-saturated.*

We do not have a complete analysis of fattenings, relating the notion to "$\mathbf{A}$-special". The author has no example of an admissible set B and a fattening C of B such that C is $\mathbf{A}$-bounded and not $\mathbf{A}$-special.

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